

# Complex Numbers in Trig Identities II

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## 1 Introduction

Trigonometry is generally considered bashy, and perhaps that is in its nature. In fact, throughout this article, I will refer to applying complex numbers into trig as a "bash", but that is not what is really meant. I truly believe that by removing all the geometric nonsense that is confused with trigonometry, which is easily done with complex numbers, one can see the elegance behind it.

Within this article, you will learn how to solve some quite advanced problems involving trigonometric identities by applying complex numbers to them. If you are not comfortable with this subject, you may consider reading "Complex Numbers in Trig Identities I".

### Prerequisites

- Know what complex numbers are and be familiar with standard/polar forms
- Have a good trig foundation: You should know how to find  $2 \cos 15 \sin 15$ .

Sample problem you should be able to do easily:

Determine  $\tan^{-1} 2 + \tan^{-1} 3$

Note: unless  $\pi$  is used, note that all angles are in degrees, despite not having degree signs.

## 2 Euler's Formula and Basic Formulas for Sin, Cos

### 2.1 An Elementary Proof of Euler's Formula

We will begin with an equation that everyone who is reading this should know:

$$e^{ix} = \cos x + i \sin x.$$

For many, this equation, known as Euler's Formula, has been memorized, but the proof has not, because the many proofs of Euler's Formula mostly require calculus. Thus, it would be nice to present an elementary proof requiring very little calculus.

**Proof:**

By definition  $e = \left(1 + \frac{z}{n}\right)^{\frac{n}{z}}$  as  $n$  approaches infinity.

Thus,  $e^z = \left(1 + \frac{z}{n}\right)^n$  as  $n$  approaches infinity. We will omit "as  $n$  approaches infinity" after this point, as it should be understood by the reader.

Now, we are trying to prove  $e^{ix} = \cos x + i \sin x$ , so it makes sense to write  $z = ix$ . Then

$e^{ix} = \left(1 + \frac{ix}{n}\right)^n$ . It's quite hard to evaluate the limit of something like this (without using circular reasoning and ending up with  $e^{ix}$ ) so we will try to find a pattern among the numbers

$$a = 1 + \frac{ix}{n}, a^2, \dots, a^n.$$

We will look at the argument and the magnitude separately. The magnitude of  $a^n$  is obviously

$$\left(1 + \frac{x}{n}\right)^{\frac{n}{2}}$$

which approaches 1 as  $n$  approaches infinity. (It's much easier to be sure of this since there isn't a pesky  $i$  term! That's the reason we separated it into magnitude and argument). We conclude that whatever the limit of  $a^n$  is, it will have magnitude 1.

Now we look at the angle: The sequence of arguments of the complex numbers  $a, a^2, \dots, a^n$  is obviously non-decreasing (at least until we go through a full unit circle, which we will disregard). The following may seem sketchy or non-rigorous, but notice that multiplication by  $a$  each time causes a point to be rotated by a certain angle.

Take a triangle with vertices  $0, 1, a$ . It is a right triangle. Now consider the triangle with vertices  $0, a, a^2$ . As  $n$  approaches infinity, we can see that this is also a right triangle.

(A  $0 - x - x$  triangle is considered right, despite being degenerate: We see it follows the pythagorean theorem).

Now the angle between  $a^2, a$  is the same as the angle between  $a, 1$ , as the ratio  $\frac{a^2}{a}$  equals the ratio  $\frac{a}{1}$ . We conclude that the triangle  $0, a, a^2$  is similar to the original triangle. They are all pretty much  $1 - 0 - 1$  degenerate right triangles. However...  $n$  never quite reaches infinity obviously. Thus, these triangles are \*almost\* degenerate, but not quite.

The degenerate side of the triangle actually has length  $\frac{x}{n}$ . Thus, the number  $a^n$  (which we already showed was on the unit circle), must be the point that we reach when we travel  $n \cdot \frac{x}{n} = x$  distance counterclockwise.

This point is clearly the same as  $\cos x + i \sin x$ , so we have proven Euler's Identity.

It is quite obvious that this proof is sketchy: However, I would rather include a sketchy, but still intuitive proof than a short proof with calculus that may be understandable, but not fully justified to the reader. The reader may find one of these rigorous proofs if he or she wishes to.

Now, we will move on. Hopefully this satisfies the reader and gives the reader some reasoning for why Euler's Formula holds.

## 2.2 Formulas for Sin, Cos

**Theorem:** Given  $z = \cos x + i \sin x$ , then

$$\begin{aligned}\cos x &= 0.5(z + \frac{1}{z}) \text{ and} \\ \sin x &= -0.5i(z - \frac{1}{z})\end{aligned}$$

**Proof:**

Notice  $\frac{1}{z} = \cos x - i \sin x$  and  $z = \cos x + i \sin x$

Thus, adding up the two equations and subtracting them from each other will yield

$z + \frac{1}{z} = 2 \cos x$ ,  $z - \frac{1}{z} = 2i \sin x$  and we obtain the desired expressions by dividing by 2 and  $2i$  respectively.

**Example:** As a reality check, make sure  $\sin^2 x + \cos^2 x = 1$

**Solution:**

$$\text{Notice } \cos^2 x + \sin^2 x = \frac{1}{4}(z^2 + 2 + \frac{1}{z^2}) - \frac{1}{4}(z^2 - 2 + \frac{1}{z^2}) = 1.$$

**Example:** Show that if  $A, B, C$  are the angles of a triangle and  $a, b, c$  are the complex numbers corresponding to them (that is,  $a = \cos A + i \sin A$ , etc.) then we have  $ab - \frac{1}{ab} = c - \frac{1}{c}$ .

**Solution:** Notice  $\sin C = \sin(180 - A - B) = \sin(A + B)$  and that  $ab = \cos(A + B) + i \sin(A + B)$  so putting it all together, we obtain  $\frac{1}{2i}(c - \frac{1}{c}) = \frac{1}{2i}(ab - \frac{1}{ab})$  so the result follows.

We will see another proof of this soon. There will be no problems for this section as we are still in the basics.

### 3 Some Applications

Not in any particular arrangement, the following problems are just to gain facility in usage of complex numbers.

We will first find expressions for two very interesting sums: Let  $z = \cos x + i \sin x$ . Find a simple closed form for

$$\cos x + \cos 2x + \dots + \cos nx$$

$$\text{and } \sin x + \sin 2x + \dots + \sin nx.$$

**Solution:**

Notice that  $\cos x, \cos 2x, \dots, \cos nx$  forms two geometric series.

That is,

$$\cos x + \cos 2x + \dots + \cos nx = 0.5(z + \frac{1}{z} + z^2 + \frac{1}{z^2} + \dots + z^n + \frac{1}{z^n}).$$

The reasoning is that  $\cos nx + i \sin nx = (\cos x + i \sin x)^n$ , by DeMoivre.

Now we can sum the series  $z + z^2 + \dots + z^n, \frac{1}{z} + \frac{1}{z^2} + \dots + \frac{1}{z^n}$  as they are geometric.

$$\text{Indeed, we may find it equals } \frac{z(z^n-1)}{z-1} + \frac{\frac{1}{z}(\frac{1}{z^n}-1)}{\frac{1}{z}-1}.$$

This is nasty! We should try to simplify it. Indeed, it simplifies to become

$$\frac{(z^n-1)(z^{n+1}+1)}{z^n(z-1)}.$$

We wish to convert all these  $z^a \pm 1$  terms into  $z^b \pm \frac{1}{z^b}$  so we can use our expressions for  $\sin x, \cos x$  so we should divide by  $z^{n+0.5}$ . However, we should not forget the original factor of  $\frac{1}{2}$ ! Putting all of these observations together, we get that the series equals

$$0.5 \cdot \frac{(z^{0.5(n+1)} + \frac{1}{z^{0.5(n+1)}})(z^{0.5n} - \frac{1}{z^{0.5n}})}{z^{0.5} - \frac{1}{z^{0.5}}}$$

$$\text{which nicely equals } \frac{\cos(\frac{x(n+1)}{2}) \sin(\frac{xn}{2})}{\sin(\frac{x}{2})}$$

By a similar method, we find

$$\sin x + \sin 2x + \dots + \sin nx = \frac{\sin \frac{nx}{2} \sin \frac{x(n+1)}{2}}{\sin \frac{x}{2}}$$

We may verify that

$$\cos x + \dots + \cos nx + i(\sin x + \dots + \sin nx) = z + z^2 + \dots + z^n$$

to confirm our answer.

**Example:**

Find  $\cos 20 \cos 40 \cos 80$ .

We present two solutions to this problem.

**Solution 1:**

Notice that if  $\cos 20 \cos 40 \cos 80 = x$ ,  $8x \sin 20 = 8 \cos 20 \sin 20 \cos 40 \cos 80$  which in turn equals  $4 \sin 40 \cos 40 \cos 80 = 2 \sin 80 \cos 80 = \sin 160$ .

We have obtained  $8x \sin 20 = \sin 160$  but as  $\sin 160 = \sin 20$ , we find  $x = \frac{1}{8}$

**With complex numbers**

Let  $z = \cos 20 + i \sin 20$  so then we must find

$$\frac{1}{8}(z + \frac{1}{z})(z^2 + \frac{1}{z^2})(z^4 + \frac{1}{z^4}).$$

Let this expression be  $x$ , so multiplying by  $z - \frac{1}{z}$ , the expression "telescopes"! Using the identity  $(z^x + \frac{1}{z^x})(z^x - \frac{1}{z^x}) = z^{2x} - \frac{1}{z^{2x}}$  we see that

$x(z - \frac{1}{z}) = 0.125(z^8 - \frac{1}{z^8})$  so then  $x \sin 20 = 0.125 \sin 160$  and the result follows.

**Remarks:** The two solutions are pretty much the same, however it is much easier to see how the product "telescopes" when you have expressions like  $x^2 + \frac{1}{x^2}, x + \frac{1}{x}$  than with expressions like  $\cos 40, \cos 20$  (at least, until you get good enough intuition to see this instantly).

This should give you a good idea of how the concept is applied.

**You cannot simply hope for plugging in these formulas and cancelling things..**

**Often, algebraic intuition is needed.**

**Problems** (solutions are in the back):

**Problem 2.1:** Solve the equation

$$\cos x + \cos 2x + \cos 3x = \sin x + \sin 2x + \sin 3x$$

**Problem 2.2:** Prove the identity  $\sin 3x(2 \cos x - 1) = \sin x(2 \cos 4x + 1)$

**Problem 2.3:** Suppose that  $\cos a + \cos b + \cos c = \sin a + \sin b + \sin c = 0$ .

Then show that

$$3 \cos(a + b + c) = \cos 3a + \cos 3b + \cos 3c \text{ and also}$$

$$3 \sin(a + b + c) = \sin 3a + \sin 3b + \sin 3c$$

## 4 Dealing with the Angles of a Triangle

Let  $A, B, C$  be the angles of a triangle with  $a = \cos A + i \sin A$  and similarly for  $b, c$ .

**Fact:**

$$\text{Then } abc = -1.$$

This should be obvious-  $abc = \cos(A + B + C) + i \sin(A + B + C) = -1$  using the property  $(\cos x + i \sin x)(\cos y + i \sin y) = \cos(x + y) + i \sin(x + y)$

**Problem from section 1:** Suppose  $A, B, C$  are the angles of a triangle and  $a, b, c$  are the complex numbers corresponding to them. Show that

$$ab - \frac{1}{ab} = c - \frac{1}{c}.$$

**New Solution:**

Multiply the fractions by  $1 = -abc$  to obtain  $ab + c = c + ab$  which is obviously true.

**Note:**

$$\text{Notice that } -1 = abc \iff ab - \frac{1}{ab} = c - \frac{1}{c}$$

This is because if  $ab - \frac{1}{ab} = c - \frac{1}{c}$  then

$$a^2 b^2 c - c = abc^2 - ab \text{ or } (abc + 1)(ab - c) = 0.$$

**Famous Identity:** Let  $A, B, C$  be the angles of a triangle. Show that

$$4 \sin A \sin B \sin C = \sin 2A + \sin 2B + \sin 2C.$$

**Solution:**

Let  $a, b, c$  be the complex numbers corresponding to  $A, B, C$ . From this point, I will use lowercase letters in this way without explanation, so keep that in mind.

Then the problem is equivalent to showing

$$(a - \frac{1}{a})(b - \frac{1}{b})(c - \frac{1}{c}) = -a^2 + \frac{1}{a^2} - b^2 + \frac{1}{b^2} - c^2 + \frac{1}{c^2}, \text{ or}$$

$$\sum_{cyc} \frac{a}{bc} - \frac{bc}{a} = \sum_{cyc} \frac{1}{a^2} - a^2$$

Now noticing  $\frac{a}{bc} = -a^2, -\frac{bc}{a} = \frac{1}{a^2}$  by  $abc = -1$  yields the result.

In reality,  $A, B, C$  don't have to be angles of a triangle for this to hold. All that is required is  $A + B + C = 180$ .

This is somewhat unrelated, but by the same method we have if

$$A + B + C = 360, \text{ then } 4 \sin A \sin B \sin C = -\sin 2A - \sin 2B - \sin 2C.$$

(Instead of  $abc = -1, abc = 1$ ).

Furthermore,  $4 \cos A \cos B \cos C = -1 - \cos 2A - \cos 2B - \cos 2C$ , a counterpart to the identity we proved. (The proof is quite similar)

We move on.

**Famous Question:** Let  $A, B, C$  be the angles of a triangle. If  $\cos 3A + \cos 3B + \cos 3C = 1$ , determine the largest angle of the triangle.

This fact also appears in 2014 AIME number 12.

The problem is quite astonishing- how do we determine an angle using just one equation involving all 3 angles? It's quite ridiculous. Regardless.. let's bash it!

**Solution:**

The condition becomes  $a^3 + b^3 + c^3 + \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = 2$  or  $a^3 + b^3 + c^3 - b^3c^3 - c^3a^3 - a^3b^3 = 2$ . Now add  $-2 = a^3b^3c^3 - 1$  to both sides and then the left side factors!

It becomes  $(a^3 - 1)(b^3 - 1)(c^3 - 1) = 0$ , implying one of  $a, b, c$  is a third root of unity! We conclude the largest angle of the triangle must be 120.

**Problem 4.1:** Given  $\sin^2 A + \sin^2 B + \sin^2 C = 2$  in a triangle, determine the largest angle of a triangle. (This one is very similar and also quite elegant!)

Until this point, we haven't covered tangent much, if at all. Thus, it is fair to prove a famous identity requiring tangent.

**Identity:** If  $\triangle ABC$  is acute, show that  $\tan a \tan b \tan c = \tan a + \tan b + \tan c$ .

**Proof:**

Obviously  $\tan A = \frac{a^2-1}{i(a^2+1)}$ . Then it remains to show

$$(a^2 - 1)(b^2 - 1)(c^2 - 1) = \sum_{cyc} (-(a^2 - 1)(b^2 + 1)(c^2 + 1)) \text{ or equivalently,}$$

$$\sum_{cyc} -a^2b^2 + b^2c^2 - a^2c^2 + b^2 + c^2 - a^2 = -a^2b^2 - b^2c^2 - c^2a^2 + a^2 + b^2 + c^2$$

Now notice that if we split the LHS into  $\sum_{cyc} -a^2b^2 + b^2c^2 - a^2c^2$  and

$\sum_{cyc} b^2 + c^2 - a^2$  that these two sums equal  $-a^2b^2 - b^2c^2 - c^2a^2$  and  $a^2 + b^2 + c^2$  respectively.

(This is because in  $\sum_{cyc} b^2 + c^2 - a^2$  each term is added twice and subtracted once, and similarly for the other sum)

Thus, the identity has been proved. So many other identities in a triangle are similarly destroyed.

**Problem 4.2:** If  $ABC$  is a triangle, show that  $\sum_{cyc} \tan \frac{A}{2} \tan \frac{B}{2} = 1$

**Problem 4.3:** If  $ABC$  is a triangle, show that  $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$ .

**Problem 4.4:** Let  $x = \sin \frac{A}{2}$  and define  $y, z$  similarly. Show that  $x^2 + y^2 + z^2 + 2xyz = 1$ .

## 5 Triangles: More Strategies (handling $R, r$ , etc.)

Quite a few trigonometric identities involve  $a, b, c, [ABC], r$ , or  $R$ . (These are the most common things that get involved.) How do we deal with these?

Denote the degree of a term as follows: if similar triangles with scale factor  $k$  make the term increase by  $k^n$ , then  $n$  is the degree. For example,  $\sin x$  always has degree 0,  $[ABC]$  has degree 2, and  $a, b, c, r, R$  have degree 1 each.

Most identities will be homogeneous- that is, all terms have the same degree. If an identity wasn't homogeneous, it would probably not be very general.

Anyway, by far the easiest way to deal with  $a, b, c$  is to write them in terms of  $2R \sin A, 2R \sin B, 2R \sin C$ . Given an identity, this is what you want to do first.

Let's try an example.

**Example:** Prove in triangle  $ABC$ ,  $a^2 \sin 2B + b^2 \sin 2A = 4[ABC]$ . (Notice all terms are degree 2).

**Step 1:** Rewrite as  $2R^2 \sin^2 A \sin 2B + 2R^2 \sin^2 B \sin 2A = 2[ABC]$ .

Now we need to deal with  $[ABC]$ - recall  $2[ABC] = R^2(\sin 2A + \sin 2B + \sin 2C)$ . This is not hard to see- simply draw  $ABC$  with circumcenter  $O$ , draw  $OA, OB, OC$ , and add up the areas of the three small triangles.

Thus it becomes showing  $2 \sin^2 A \sin 2B + 2 \sin^2 B \sin 2A = \sin 2A + \sin 2B + \sin 2C$ . Now our conventional method apply- complex numbers and expansion immediately turns it into

$$-a^2 b^2 + b^2 + a^2 + \frac{1}{a^2 b^2} - \frac{1}{a^2} = \sum_{cyc} a^2 - \frac{1}{a^2}$$

which is true by rewriting  $\frac{1}{a^2 b^2}$  as  $c^2$  and  $-a^2 b^2$  as  $-\frac{1}{c^2}$ .

So how did the intuition for this occur? Well, here is a tip:

**Convert everything in terms of trigonometric functions and  $R$  when possible.**

You generally convert  $a = 2R \sin A$  first as this requires no thinking. The next step is trickier:

Rewrite  $[ABC]$  as either  $0.5R^2(\sin 2A + \sin 2B + \sin 2C)$  or  $2R^2 \sin A \sin B \sin C$ . This doesn't seem like a big deal, but it can make a massive difference in the amount of bashing you do after converting to complex numbers.

What about  $r$ , you might ask? There are also two ways to go about doing this.

The first way is to notice  $r = (s - a) \tan \frac{A}{2}$ . Now rewrite  $s - a = \frac{b+c-a}{2} = R \sin B + R \sin C - R \sin A$ .

The other way is to use  $r = \frac{[ABC]}{s}$  and now we have already covered how to deal with  $[ABC]$ ,  $s$ .

Let's see these methods at work in more problems.

**Example:** Prove that  $\sum_{cyc} a^3 \cos(B - C) = 3abc$

**Solution:**

Noting  $a = 2R \sin A$ , etc. and dividing  $8R^3$  seems like a good step. We obtain

$\sum_{cyc} \sin^3 A \cos(B - C) = 3 \sin A \sin B \sin C$ . Now applying complex numbers

and noting  $\cos(B - C) = 0.5(\frac{b}{c} + \frac{c}{b})$  gives us some nice stuff:

$$\sum_{cyc} (a^3 - 3a + \frac{3}{a} - \frac{1}{a^3})(\frac{b}{c} + \frac{c}{b}) = 6 \sum_{cyc} \frac{c}{ab} - \frac{ab}{c}.$$

Now notice  $(a^3 - 3a + \frac{3}{a} - \frac{1}{a^3})(\frac{b}{c} + \frac{c}{b}) = \frac{a^3b}{c} + \frac{a^3c}{b} - \frac{3ab}{c} - \frac{3ac}{b} + \frac{3b}{ac} + \frac{3c}{ab} - \frac{b}{a^3c} - \frac{c}{a^3b}$ .

For some cancellations, notice

$$\sum_{cyc} -\frac{3ab}{c} - \frac{3ac}{b} + \frac{3b}{ac} + \frac{3c}{ab} = 6 \sum_{cyc} -\frac{ab}{c} + \frac{c}{ab}.$$

Then it remains to show

$$\sum_{cyc} \frac{a^3b}{c} + \frac{a^3c}{b} - \frac{b}{a^3c} - \frac{c}{a^3b} = 0$$

which becomes  $\sum_{cyc} -a^4b^2 - a^4c^2 + b^4c^2 + b^2c^4 = 0$  which is obvious.

This problem was, admittedly, bashy, which is why algebraic intuition is needed. Sometimes, it is helpful to switch between  $\sum_{cyc} a^4b^2 + a^2b^4$  and notation

like  $(4, 2, 0)$  because both have their advantages.

However, I will not use  $(4, 2, 0)$  notation here.

**Example:** Show that  $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$ . This is useful for taking  $r$  out of problems where it is annoying.

**Solution:**

Rewrite  $r = (s - a) \tan \frac{A}{2}$  and then rewrite  $a, b, c$  in  $s - a$  using  $R$  and Law of Sines. Then the identity becomes  $(\sin B + \sin C - \sin A) = 4 \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$  after rewriting  $\tan \frac{A}{2}$ .

Applying complex numbers and expanding, we need to show

$$b - \frac{1}{b} + c - \frac{1}{c} - a + \frac{1}{a} = -i(\sqrt{abc} + \frac{1}{\sqrt{abc}} - \sqrt{\frac{ab}{c}} - \sqrt{\frac{ac}{b}} + \sqrt{\frac{bc}{a}} - \sqrt{\frac{c}{ab}} - \sqrt{\frac{b}{ac}} + \sqrt{\frac{a}{bc}}).$$

And clearly  $\sqrt{abc} + \frac{1}{\sqrt{abc}} = i - i = 0$ . Furthermore, write all the terms of

the form  $\sqrt{\frac{ab}{c}}$  as  $\sqrt{-\frac{1}{c^2}} = \frac{i}{c}$  and the same for terms in the form  $\sqrt{\frac{c}{ab}}$  and then it becomes showing

$$b - \frac{1}{b} + c - \frac{1}{c} - a + \frac{1}{a} = b - \frac{1}{b} + c - \frac{1}{c} - a + \frac{1}{a} \text{ which is clearly true.}$$

**Problem 5.1:** Show that  $2R \sin A \sin B \sin C = r(\sin A + \sin B + \sin C)$ .

**Problem 5.2:** Show that  $a \cos A + b \cos B + c \cos C = \frac{abc}{2R^2}$

**Problem 5.3:** Show that  $s = 4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

## 6 Conclusion and Remarks

### Remarks

First, it is important to note that **complex numbers have some serious flaws when it comes to bashing trig.**

There is one particular area where this flaw is apparent: inequalities.

Basic inequalities like  $\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \leq 0.125$  are incredibly hard to engineer with complex numbers, because you cannot apply inequalities with complex numbers (unless of course it is the triangle inequality). Thus, we would have to simplify the LHS into a real expression first, and THEN show it is less than one eighth.

Now, several things must be re-stressed over and over.

Firstly, the complex numbers method typically **requires algebraic intuition.**

From the problems we have shown solutions to, this is not as apparent, but for certain problems in the problem section coming up, bashing and chugging simply won't work. A simple tweaking of complex numbers can drastically increase or decrease the amount of bash you need to do, and drastically change the elegance of your solution. Furthermore, one must be very neat in their expansions. All problems thus far have generally been symmetric or cyclic in A,B,C but as you will see, this is not always the case.

Secondly, it is important to recognize trigonometric identities hidden inside complex numbers. When you see a massive chunk like  $0.25 \prod (a^2 + b^2c^2 - 2)$ , what do you think of first?

After experience, you'll instantly see it equals  $0.25 \prod (a^2 + \frac{1}{a^2} - 2)$  which, if you notice the perfect square that is hidden, becomes  $(4 \sin A \sin B \sin C)^2$  (or perhaps the form  $(\sin 2A + \sin 2B + \sin 2C)^2$  is more useful? who knows?)

Lastly, not everything should be bashed. Oops, we said this already.

After this, you will find a list of identities/formulas with their appropriate proofs. Then, there will be some review/challenge problems. Finally, there is a solutions manual with all the solutions to all the problems, whether they were in the sections or the problems list.

**I would like to thank AkshajK** for advising me many many times in creating this, whether it was deleting/adding things to the table of contents (which is not included in here) or other random stuff. Also, I took some problems from "53 Trigonometry Problems" compiled by Amir Hossein and from "103 Trigonometry Problems". You may look in there for further problems, but many have already been used in creating this.

## 7 Problems

Enjoy the problems! They are arranged in no particular order. This is so that you try every single problem instead of skipping ones that are easy/hard.

### 7.1 Regular Problems

1. Prove that  $\tan \frac{\pi}{7} \tan \frac{2\pi}{7} \tan \frac{3\pi}{7} = \sqrt{7}$
2. Suppose  $x, y, z, p$  satisfy  $p(\cos(x+y+z)) = \cos x + \cos y + \cos z$  and similarly for  $\sin$ . Prove that  $\cos(x+y) + \cos(y+z) + \cos(z+x) = p$
3. Prove  $\tan \frac{\pi}{13} + 4 \sin \frac{4\pi}{13} = \tan \frac{3\pi}{13} + 4 \sin \frac{3\pi}{13}$
4. Evaluate  $\sin \theta + 0.5 \sin 2\theta + 0.25 \sin 3\theta + \dots$
5. Let  $a, b$  be angles of a scalene triangle. Show that the conditions  $a + b = 90$ ,  $(\cos^2 a + \cos^2 b)(1 + \tan a \tan b) = 2$  are equivalent.
6. In a triangle ABC, show that  $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
7. a) For  $x$  not a multiple of 30, show that  $\frac{\tan 3x}{\tan x} = \tan(60-x) \tan(60+x)$   
...b) Show that  $\tan(x-72) \tan(x-36) \tan x \tan(x+36) \tan(x+72) = \tan 5x$ .
8. Prove in triangle ABC,  $\frac{a-b}{a+b} = \tan \frac{A-B}{2} \sin \frac{C}{2}$
9. Prove a triangle ABC is isosceles iff  $a \cos B + b \cos C + c \cos A = \frac{a+b+c}{2}$

### 7.2 Olympiad Problems

10. (IMO 1962) Solve the equation in  $(0, 2\pi)$  of  $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$
11. (IMO 1966) Prove that for every natural number  $n$ , and for every real number  $x \neq \frac{k\pi}{2^t}$  ( $t = 0, 1, \dots, n$ ;  $k$  any integer)

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \dots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x$$

12. (IMO 1996) Show that if in triangle ABC,  $a + b = \tan \frac{C}{2}(a \tan A + b \tan B)$ , then triangle ABC is isosceles.
13. (ISL) Prove that for all integers  $n$  the following is true:  
 $2^n \prod_{k=1}^n \sin \frac{k\pi}{2^{n+1}} = \sqrt{2n+1}$
14. (ISL) Given real  $a$  and integer  $m > 0$ , and  $P(x) = x^{2m} - 2|a|^m x^m \cos \theta + a^{2m}$ , factorize  $P(x)$  as a product of  $m$  real quadratic polynomials
15. (1985 Iran) Let  $\alpha$  be an angle such that  $\cos \alpha = \frac{p}{q}$ , where  $p$  and  $q$  are two integers. Prove that the number  $q^n \cos n\alpha$  is an integer.
16. (ISL) Show that a triangle whose angles  $A, B, C$  satisfy the equality

$$\frac{\sin^2 A + \sin^2 B + \sin^2 C}{\cos^2 A + \cos^2 B + \cos^2 C} = 2$$

is a rectangular triangle.

17. (1980 USAMO) Let  $F_r = x^r \sin rA + y^r \sin rB + z^r \sin rC$ , where  $x, y, z, A, B, C$  are real and  $A + B + C$  is an integral multiple of  $\pi$ . Prove that if  $F_1 = F_2 = 0$ , then  $F_r = 0$  for all positive integral  $r$ .

18. (1996 USAMO) Prove that the average of the numbers  $n \sin n^\circ$  ( $n = 2, 4, 6, \dots, 180$ ) is  $\cot 1^\circ$

That should be plenty for one to practice on. If you want more practice, as we have already stated, many problems were taken out of 103 Trigonometry Problems and 53 Trigonometry Problems, so those may not be good sources. ISL and USAMO lists will not be good sources either, because I have used many of their trigonometry problems that are not inequalities in here.

## 8 Buffer Zone

This entire almost-blank page is being used to separate the problems and the solutions, so that you do not accidentally scroll down and see the solutions. The next page will contain solutions to problems from previous sections, not the problem section.

## 9 Solutions to Problems from Previous Sections

**Solution 2.1:** Rewriting in terms of complex numbers and expanding yields (defining  $a = \operatorname{cis} x$ ) that

$$a^6(1-i) + a^5(1-i) + a^4(1-i) - x^2(1+i) - x(1+i) - (1+i) = 0 \text{ or obviously} \\ (a^2 + a + 1)(a^4(1-i) - (1+i)) = 0 \text{ and then it is apparent that} \\ (a^2 + a + 1)(a^4 - i) = 0. \text{ Then obviously } a = 120 + 360k, 240 + 360k, 22.5 + 90k.$$

**Solution 2.2:** Using  $a = \cos x + i \sin x$  yields

$(a^3 - \frac{1}{a^3})(a^2 + \frac{1}{a^2} - 1) = (a - \frac{1}{a})(a^4 + \frac{1}{a^4} + 1)$ . Let us not expand. Rather, multiply both sides by  $a + \frac{1}{a}$ . It is evident that the LHS becomes

$$(a^3 - \frac{1}{a^3})(a^3 + \frac{1}{a^3}) = x^6 - \frac{1}{x^6}.$$

The RHS is obviously  $(x^2 - \frac{1}{x^2})(x^4 + \frac{1}{x^4} + 1)$  which is just the difference of cubes factorization in disguise. Truly, it becomes  $x^6 - \frac{1}{x^6}$ .

**Solution 2.3:** We easily obtain  $a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = a + b + c - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} = 0$ . Thus,  $a + b + c = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$ .

Then we conclude  $a^3 + b^3 + c^3 = 3abc, \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} = \frac{3}{abc}$ .

And by subtracting the two equations together and adding them together, we obtain the desired result(s).

**Solution 3.1:** Substitute  $w = \cos \frac{\pi}{2n+1} + i \sin \frac{\pi}{2n+1}$ .

Then clearly  $w^{2n+1} = -1, w^{4n+2} = 1$ .

Now notice  $2^{2m}(\cos^{2m} \frac{k\pi}{2n+1}) = (w^k + \frac{1}{w^k})^{2m}$ .

By binomial expansion, this becomes  $w^{2k} + \binom{2m}{1}w^{2k-4} + \dots + \frac{1}{w^{2k}}$ .

Now if we expand each of these  $(w^k + \frac{1}{w^k})^{2m}$  terms and group everything by its binomial coefficient, something interesting happens.

First, notice the constant terms all equal  $\binom{2m}{m}$  and there are  $n$  of them, yielding  $n\binom{2m}{m}$ .

Now notice we have many geometric series of the form

$$\binom{2m}{k}(w^{n(2m-2k)} + w^{(n-1)(2m-2k)} + \dots + \frac{1}{w^{n(2m-2k)}}).$$

However, notice that none of them have a constant term! Thus, we add a constant term equal to the binomial coefficient into each geometric series.

Of course, a total of  $\binom{2m}{0} + \binom{2m}{1} + \dots + \binom{2m}{m-1}$  is added to "complete" each geometric series, and thus this quantity must be subtracted from  $n\binom{2m}{m}$ .

Now notice each geometric series of the form

$$w^{n(2m-2k)} + w^{(n-1)(2m-2k)} + \dots + \frac{1}{w^{n(2m-2k)}} \text{ is equal to } 0! \text{ This is from}$$

applying the geometric series formula.

We conclude, after a long expansion process, that  $\sum \cos^{2m} \frac{k\pi}{2n+1}$  is equal to  $\frac{1}{2^{2m}}(n\binom{2m}{m} - \binom{2m}{m-1} - \dots - \binom{2m}{0})$ .

(Remember that we got rid of the  $2^{2m}$  term a long time ago. Also, the last expression may be simplified.)

**Solution 3.2:** This one is quite easy, so we shall not go into detail. Define  $w = \operatorname{cis} 6$  and then  $w^{15} = i, w^{30} = -1, w^{60} = 1$ . Now make everything in terms of  $w$ , and expand.

$$\text{You should get } \frac{2z}{z^2+1} - \frac{2iz}{z^4-1} + \frac{2iz^4}{z^8-1} + \frac{2iz^8}{z^{16}-1} = 0.$$

Notice the denominators are factors of  $z^{16} - 1$ . Thus, multiply the equation by this factor and it should reduce to a trivial expression (with the help of  $z^{15} = i$ ).

**Solution 3.3:**

We don't like the square root, so let's prove the equivalent

$$\sin \frac{\pi}{2n} \sin \frac{2\pi}{2n} \dots \sin \frac{(2n-1)\pi}{2n} = \frac{n}{2^{2n-2}}$$

(I multiplied each  $\sin x$  term by  $\sin(180 - x)$  which is equal to it. Also, there is a  $1 = \sin \frac{n\pi}{2n}$  term inside.)

Now, we define  $w = \text{cis} \frac{\pi}{2n}$  so that  $w^n = i, w^{2n} = -1, w^{4n} = 1$ .

Then we express everything in  $w$ : Grouping all the  $2i$  terms together, the LHS is equivalent to

$$\frac{1}{(2i)^{2n-1}} \left(w - \frac{1}{w}\right) \left(w^2 - \frac{1}{w^2}\right) \dots \left(w^{2n-1} - \frac{1}{w^{2n-1}}\right) \text{ which in turn equals}$$

$$\frac{1}{(2i)^{2n-1} w^{n(2n-1)}} \left(1 - \frac{1}{w^2}\right) \left(1 - \frac{1}{w^4}\right) \dots \left(1 - \frac{1}{w^{4n-2}}\right).$$

Remembering that  $w^n = i$ , we see that the  $\frac{1}{(2i)^{2n-1} w^{n(2n-1)}}$  is equal to  $\frac{1}{(2)^{2n-1}}$ .

Now we multiply both sides by  $2^{n-1}$  and the problem becomes showing

$$\left(1 - \frac{1}{w^2}\right) \left(1 - \frac{1}{w^4}\right) \dots \left(1 - \frac{1}{w^{4n-2}}\right) = 2n. \text{ Rewrite this as}$$

$$\left(1 - w^{4n-2}\right) \left(1 - w^{4n-4}\right) \dots \left(1 - w^2\right).$$

Now we prove the following: If  $P(x) = x^{2m-2} + x^{2m-4} + \dots + 1$ , then  $P(x)$  can be factored as  $(x^2 - w^2)(x^2 - w^4)(x^2 - w^6) \dots (x^2 - w^{2m-2})$  where  $w$  is a primitive  $2m$ th root of unity.

This is quite obvious, because when  $x^2$  is equal to  $w^2, w^4, \dots, w^{2m-2}$  both sides are 0. (This is because  $P(x)$  is equal to  $\frac{x^{2m}-1}{x^2-1}$ .)

Returning to the problem, we set  $x^2 = 1, m = 2n$ , then we see that

$$\left(1 - w^{4n-2}\right) \left(1 - w^{4n-4}\right) \dots \left(1 - w^2\right) = 1^{4n-2} + 1^{4n-4} + \dots + 1^0 = 2n. \text{ Thus,}$$

since all steps are reversible, we are done.

**Solution 3.4:** This one is not hard. Define  $w = \text{cis} \frac{\pi}{8}$ . Evidently  $w^4 = i, w^8 = -1, w^{16} = 1$ .

Then expanding trivializes the problem. We proceed as in solution 3.1 by "completing the geometric series" for each binomial coefficient.

**Solution 4.1:** Expanding everything, we obtain  $a^2 + b^2 + c^2 + a^2b^2 + b^2c^2 + c^2a^2 = -2$ .

Now add  $2 = a^2b^2c^2 + 1$  to both sides, and it becomes  $(a^2+1)(b^2+1)(c^2+1) = 0$  so one of  $a, b, c$  is equal to  $i$  and one of the angles equals 90 degrees.

**Solution 4.2:** Notice  $\tan \frac{A}{2} = \frac{a-1}{i(a+1)}$ .

$$\text{Then } \sum_{cyc} \tan \frac{A}{2} \tan \frac{B}{2} = \sum_{cyc} -\frac{(a-1)(b-1)}{(a+1)(b+1)}.$$

Then we need to show  $\sum_{cyc} -(a-1)(b-1)(c+1) = (a+1)(b+1)(c+1)$  or

$$\sum_{cyc} -abc - ab + ac + a + bc - c + b - 1 = ab + bc + ca + a + b + c.$$

Now notice in the LHS, each  $ab$  term appears twice as positive and once as negative. The same is true of each  $a$  term. The  $-abc$  and  $-1$  terms just cancel each other out. Then it is evident that both sides are equal.

**Solution 4.3:** This one is also trivial by expansion. We need to show

$$\sum_{cyc} -(a^2+1)(b^2+1)(c^2-1) = (a^2-1)(b^2-1)(c^2-1) \text{ and then we expand}$$

both sides and compare the coefficients of each term.

**Solution 4.4:** After expansion it becomes

$$\sum_{cyc} i \left(a + \frac{1}{a} - 2\right) - \sqrt{\frac{bc}{a}} - \sqrt{\frac{a}{bc}} = -6i$$

Now rewrite  $\frac{bc}{a} = -\frac{1}{a^2}$  and then  $\sqrt{\frac{bc}{a}} = \frac{i}{a}$  and same for all the others. Meanwhile,  $\sqrt{\frac{a}{bc}} = \frac{a}{i}$ .

Then lots of things cancel, leaving us with  $\sum_{cyc} -2i = -6i$ .

**Solution 5.1:** We already showed that  $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$ .

Substitute that in, and we get  $\sin A \sin B \sin C = 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} (\sin A + \sin B + \sin C)$ .

Now rewrite on the LHS  $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$ . Then we cancel out all the sines to obtain

$$4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \sin A + \sin B + \sin C$$

And it is not hard to see that this is true by expansion. (Once again, we use  $\sqrt{\frac{bc}{a}} = \frac{i}{a}$ ,  $\sqrt{\frac{a}{bc}} = \frac{a}{i}$ .)

**Solution 5.2:** This is not really a complex number proof, but we use an identity that we proved earlier with complex numbers.

Recall that  $4 \sin A \sin B \sin C = \sin 2A + \sin 2B + \sin 2C$ . (refer to section 4)

In the problem, once you substitute  $a = 2R \sin A$ ,  $b = 2R \sin B$ ,  $c = 2R \sin C$  and cancel out all the  $R$ s,

it boils down to  $4 \sin A \sin B \sin C = \sin 2A + \sin 2B + \sin 2C$  which is true.

**Solution 5.3:** Substitute  $s = R(\sin A + \sin B + \sin C)$ . Then we need to prove

$$\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

Applying complex numbers and expanding, we need to show that

$$\sum_{cyc} a - \frac{1}{a} = i(\sqrt{abc} + \frac{1}{\sqrt{abc}} + \sum_{cyc} (\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}})).$$

The RHS is equal to  $\sum_{cyc} -\frac{1}{a} + a$  because  $\sqrt{abc} + \frac{1}{\sqrt{abc}} = 0$ .

We now present solutions to problems from the problems section on the next page.

## 10 Solutions to Problems

### 10.1 Regular Problem Solutions

**Solution 1a:** We prove the equivalent  $\tan \frac{\pi}{7} \tan \frac{2\pi}{7} \dots \tan \frac{6\pi}{7} = -7$ .

Let  $w$  be a primitive 14th root of unity, then we need to prove

$$\frac{(w^2-1)(w^4-1)\dots(w^{12}-1)}{(w^2+1)(w^4+1)\dots(w^{12}+1)} = 7$$

**Lemma:** Let  $w$  be a  $4k-2$ th root of unity. The polynomial  $P(x) = (x^2 - w^2)(x^2 - w^4)\dots(x^2 - w^{4k-2})$  is equivalent to the polynomial  $x^{4k-4} + x^{4k-6} + \dots + 1$ .

The proof of this fact is simple. In fact, we already proved it earlier in a solution to a different problem (3.2).

Now we apply the lemma, noting  $\frac{(w^2-1)(w^4-1)\dots(w^{12}-1)}{(w^2+1)(w^4+1)\dots(w^{12}+1)}$  is equal to  $\frac{P(1)}{P(i)}$  (where  $k=4$ ) and this is easily evaluated as  $\frac{7}{1}$ .

**Solution 1b:** The proof is evident by using our lemma for a general  $k$ .

**Solution 2:** We obtain  $pxyz + \frac{p}{xyz} = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

Also,  $pxyz - \frac{p}{xyz} = x + y + z - \frac{1}{x} - \frac{1}{y} - \frac{1}{z}$  and thus

$$pxyz = x + y + z, \frac{p}{xyz} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, p = xy + yz + zx.$$

We need to show  $xy + yz + zx + \frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx} = 2p$  which is now obvious.

**Solution 3:** This is quite forward by setting  $w = \text{cis} \frac{\pi}{13}$  and expanding.

**Solution 4:** Define  $z = \text{cis} \theta$  and split it into two geometric series:

$\frac{z}{2i}(1 + \frac{z}{2} + \frac{z^2}{4} + \dots) - \frac{1}{2zi}(1 + \frac{1}{2z} + \dots)$  and evaluate them to be  $\frac{2z^2-2}{i(-2z^2+5z-2)}$  and now divide by  $z$  to obtain

$$\frac{2(z-\frac{1}{z})}{-2i(z+\frac{1}{z})+5i} \text{ and this is equal to } \frac{4 \sin \theta}{5-4 \cos \theta}.$$

**Solution 5:** Clearly if  $a+b=90$ , the relation is satisfied. Thus, assume the relation is satisfied, and we will prove  $a+b=90$ .

With complex numbers it becomes

$$(a^2 + b^2 + 4 + \frac{1}{a^2} + \frac{1}{b^2})(a^2 + b^2) = 4(a^2 + 1)(b^2 + 1)$$

or  $a^4 + 1 + b^4 + 1 + \frac{a^2}{b^2} + \frac{b^2}{a^2} = 2a^2b^2 + 4$ . If we are clever with grouping, we may see

$$(a^4 + b^4) + \frac{a^4+b^4}{a^2b^2} - 2a^2b^2 - 2 = 0 \text{ or}$$

$$(a^4 + b^4)(1 + \frac{1}{a^2b^2}) - 2a^2b^2(1 + \frac{1}{a^2b^2}) = 0 \text{ and then obviously}$$

$$(a^2 - b^2)^2(1 + \frac{1}{a^2b^2}) = 0 \text{ so either}$$

$a^2 = b^2, ab = i$ . Since  $a, b$  are both less than 180 in angle measure,  $a^2 = b^2 \implies a = b$

But we are given that the triangle is scalene, so  $a = b$  is impossible and we know  $ab = i$ , so  $A + B = 90$

**Solution 6:** Immediately apply complex numbers to obtain

$$a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 2 - i \prod_{cyc} (\sqrt{a} - \frac{1}{\sqrt{a}}).$$

This must be the third or fourth time you've encountered  $(\sqrt{a} - \frac{1}{\sqrt{a}})$  and the equivalence of the LHS, RHS should be obvious.

**Solution 7a:** define  $w = \text{cis} 60$  and we need to show

$$\frac{x^6-1}{x^6+1} = \frac{(x^2-1)(x^2-w^4)(x^2-w^2)}{(x^2+1)(x^2+w^4)(x^2+w^2)}.$$

Now it should be obvious that  $(x^2-1)(x^2-w^2)(x^2-w^4) = x^6-1, (x^2+1)(x^2+w^2)(x^2+w^4) = x^6+1$  and so we are done.

**Solution 7b:** We proceed as in the previous problem except we define  $w = \text{cis}36$ . Everything else follows the same steps.

It should be noted that 7a, 7b can easily be generalized.

**Solution 8:** It is equivalent to

$$(a - \frac{1}{a} - b + \frac{1}{b})(a + b) = -(a - \frac{1}{a} + b - \frac{1}{b})(a - b)(\frac{c-1}{c+1})$$

Notice we can pull an  $a - b$  factor out of  $a - b - \frac{1}{a} + \frac{1}{b}$  and an  $a + b$  factor out of  $a + b - \frac{1}{a} - \frac{1}{b}$ .

Then we are left with  $(1 + \frac{1}{ab})(c + 1) = -(c - 1)(1 - \frac{1}{ab})$  which, employing  $abc = -1$ , is trivial.

**Solution 9** Let us prove that if it is isosceles, the condition holds. WLOG  $A = B$ . Then we may notice  $c = 2a \cos A$  and the relation becomes trivial. It reduces to  $\cos 2A = 2 \cos^2 A - 1$ .

The reverse direction is tougher. Expansion yields

$$\sum_{cyc} (ab - \frac{b}{a} + \frac{a}{b} - \frac{1}{ab}) = \sum_{cyc} a - \frac{1}{a}.$$

Noting  $ab = -\frac{1}{c}$ ,  $\frac{1}{ab} = -c$ , we are left with

$$\sum_{cyc} \frac{a}{b} - \frac{b}{a} = 0.$$

Clearing denominators yields  $a^2c + b^2a + c^2b - b^2c - c^2a - a^2b = 0$ . Now, if you see a clever factorization, everything is over. Adding  $abc - abc = 0$  to both sides, we may now factor it easily as

$$(b - a)(c - b)(c - a) = 0 \text{ and then it is apparent that the triangle is isosceles.}$$

## 10.2 Solutions to Olympiad Problems

**Solution 10:** Denote  $w = \text{cis}x$  and then it follows that

$$w^2 + \frac{1}{w^2} + w^4 + \frac{1}{w^4} + w^6 + \frac{1}{w^6} = -2.$$

Substitute  $z = w^2 + \frac{1}{w^2}$  to obtain

$$z + z^2 - 2 + z^3 - 3z = -2 \text{ or } z^3 + z^2 - 2z = 0 \text{ implying } z = 0, -2, 1.$$

First we handle  $z = -2$  as it is easy. It implies  $(w^2 - 1)^2 = 0$ ,  $w = \pm 1$ . Then  $x = 180$

Now we handle  $z = 0$ . It implies  $w^4 + 1 = 0$ ,  $w = \text{cis}45, 135, 225, 315$  (and those are the values of  $x$ ).

Finally, consider  $z = 1$  or  $w^4 - w^2 + 1 = 0$ , implying  $w^2 = \text{cis}60, 300$ .

This means that  $w = \text{cis}30, 150, 210, 330$ .

We conclude  $x = 30, 45, 135, 150, 180, 210, 225, 315, 330$

**Solution 11:** This one does not need complex numbers, as induction trivializes it, but we may attempt complex numbers regardless.

With  $w = \text{cis}x$  we obtain

$$\frac{x^2}{x^4-1} + \frac{x^4}{x^8-1} + \dots + \frac{x^{2^n}}{x^{2^{n+1}}-1} = \frac{1}{x^2-1} - \frac{1}{x^{2^{n+1}}-1}.$$

But it telescopes! Notice  $\frac{x^{2^n}}{x^{2^{n+1}}-1} + \frac{1}{x^{2^{n+1}}-1}$  is equal to  $\frac{1}{x^{2^n}-1}$  and this process keeps on going until we are left with

$$\frac{1}{x^2-1} = \frac{1}{x^2-1}. \text{ Thus the identity is proven.}$$

**Solution 12:** This is probably the ugliest problem you will see in this article. However, it should teach you not to fear messes and that with courage, one can bash almost anything.

Standard notation and substituting  $a = 2R \sin A$ ,  $b = 2R \sin B$  yields

$$-a + \frac{1}{a} - b + \frac{1}{b} = \frac{c-1}{c+1} \left( \frac{(a^2-1)^2}{a^3+a} + \frac{(b^2-1)^2}{b^3+b} \right)$$

Now we get rid of  $c$  and simplify a little:

$$(ab-1)(a+b) = \frac{ab+1}{ab-1} \left( \frac{b(a^2-1)^2}{a^2+1} + \frac{a(b^2-1)^2}{b^2+1} \right)$$

Here goes to expansion:

$$(ab-1)(a+b)(ab-1)(a^2+1)(b^2+1) = (ab+1)(b(b^2+1)(a^2-1)^2 + a(a^2+1)(b^2-1)^2)$$

Let's expand the right side first, but keep the  $ab+1$  term as it is.

$$\sum b^3a^4 + ba^4 - 2a^2b^3 - 2a^2b + b^3 + b$$

where the summation goes through  $a, b$ .

We may factor out of the sum an  $a+b$  term to obtain

$$(a+b)(a^3b^3 + a^3b + ab^3 - 3a^2b^2 - 3ab + a^2 + b^2 + 1)$$

Now dividing out an  $a+b$  term from both sides, it is finally time to expand.

$$(a^2b^2 - 2ab + 1)(a^2b^2 + a^2 + b^2 + 1) = (ab+1)(a^3b^3 + a^3b + ab^3 - 3a^2b^2 - 3ab + a^2 + b^2 + 1)$$

Because the expansion is in two variables, it is not difficult to compute.

In reality, we are only computing  $12 + 16 = 28$  products, not too bad.

$$\text{Neat and flawless expansion yields } -4a^3b + 8a^2b^2 - 4ab^3 = 0 \text{ or } -4ab(a-b)^2 = 0.$$

As  $a, b$  are representations of angles of a triangle,  $a = b$  so the triangle is isosceles.

### Problem 13:

Denote  $w = \text{cis} \frac{\pi}{2n+1}$  so that  $w^{2n+1} = -1, w^{4n+2} = 1$ .

Then we wish to show  $i^n \sqrt{2n+1} = (w - \frac{1}{w})(w^2 - \frac{1}{w^2}) \dots (w^n - \frac{1}{w^n})$ .

This is equivalent to

$$(-1)^n (2n+1) = (w - \frac{1}{w})^2 (w^2 - \frac{1}{w^2})^2 \dots (w^n - \frac{1}{w^n})^2.$$

Or we can express this as

$$(-1)^n (2n+1) = (w - \frac{1}{w}) \dots (w^n - \frac{1}{w^n}) (w^{n+1} - \frac{1}{w^{n+1}}) \dots (w^{2n} - \frac{1}{w^{2n}}).$$

This is because  $w^n - \frac{1}{w^n} = w^{n+1} - \frac{1}{w^{n+1}}$  and so on.

At this point you should know how to proceed. First, we multiply both sides by a bunch of  $w$ s to obtain

$$(-1)^n (2n+1) w^{n(2n+1)} = (w^2 - 1)(w^4 - 1) \dots (w^{4n} - 1).$$

Notice the LHS equals  $2n+1$ .

Furthermore, you should be familiar with the right hand side:

$(w^2 - x)(w^4 - x) \dots (w^{4n} - x) = x^{2n} + x^{2n-1} + \dots + 1$ . (This is the third or fourth encounter of this useful fact).

Plugging in  $x = 1$ , we obtain  $2n+1 = 2n+1$ , so the problem is solved.

**Solution 14:** This doesn't really need the techniques we have learned, but it is quite a nice problem regardless.

Setting  $x^{2m} - 2|a|^m x^m \cos \theta + a^{2m} = 0$  and viewing it as a quadratic in  $x^m$ , we obtain that

$x^m = |a|^m (\cos \theta \pm i \sin \theta)$  by the quadratic formula.

Then say  $w = \cos \frac{\theta}{m} + i \sin \frac{\theta}{m}, z = \text{cis} \frac{2\pi}{m}$ , so that

$x = |a|w, |a|wz, |a|wz^2, \dots, |a|wz^{m-1}$  or

$x = |a|\bar{w}, |a|\bar{w}z, \dots, |a|\bar{w}z^{m-1}$ .

Thus, we may group the factors of the form  $(x - |a|wz^k), (x - |a|\bar{w}z^k)$  together and obtain factors of the form  $(x^2 - 2|a| \cos \frac{\theta + 2k\pi}{m} + a^2)$ .

**Solution 15:** We proceed with strong induction on  $n$ . The cases  $n = 1, n = 2$  are trivial.

Let  $a$  be the complex number corresponding to  $\alpha$ . Then  $a + \frac{1}{a} = \frac{2p}{q}$ .

We wish to show  $\frac{q^n}{2}(a^n + \frac{1}{a^n})$  is an integer.

Define  $T_n = \frac{q^n}{2}(a^n + \frac{1}{a^n})$ .

Then  $T_n = 2T_{n-1}T_1 - q^2T_{n-2}$ , and we have proven that  $T_1, T_2, \dots, T_{n-1}$  are integers. Immediately, we see  $T_n$  is an integer.

Complex numbers helped us see very quickly that  $T_n = 2T_{n-1}T_1 - q^2T_{n-2}$ , which, if we had stayed with  $\cos$ , would have been tough to see.

**Solution 16:**

It is not hard to see  $\sin^2 A + \sin^2 B + \sin^2 C = 2(\cos^2 A + \cos^2 B + \cos^2 C)$  or  $\sin^2 A + \sin^2 B + \sin^2 C = 2$  which is equivalent to

$$a^2 + b^2 + c^2 + a^2b^2 + b^2c^2 + c^2a^2 = -2 \text{ implying}$$

$$a^2b^2c^2 + a^2b^2 + b^2c^2 + c^2a^2 + a^2 + b^2 + c^2 + 1 = 0 \text{ or}$$

$(a^2 + 1)(b^2 + 1)(c^2 + 1) = 0$  implying one of  $a, b, c$  is  $i$ , so it has a measure of 90 degrees.

**Solution 17:**

Substitute  $d = xcisA, e = ycisB, f = zcisC$ .

Notice that since  $A + B + C$  is a multiple of  $\pi$ ,  $def = \pm xyz$ .

Now suppose we define  $g_n = d^n + e^n + f^n$ .

If we show that  $g_n$  is real for integers  $n$ , then  $g_n$ 's imaginary part, which is  $F_r$  will be zero. We may attempt to show this now.

We have shown  $def = \pm xyz$  which is real.

We are given  $F_1 = 0$ , implying that  $d + e + f$  is real.

Finally,  $0.5(g_1^2 - g_2) = de + ef + fd$  is real.

Thus, there exists a cubic  $x^3 + a_2x^2 + a_1x + a_0$  with all real coefficients with roots  $d, e, f$ . (The whole purpose of showing the symmetric sums of  $d, e, f$  were real was to use Vieta's Formulas to construct a cubic).

Therefore, if we plug in  $d, e, f$  into this cubic we get

$$d^3 + a_2d^2 + a_1d + a_0 = 0.$$

$$\text{Multiplying gives } d^{n+3} + a_2d^{n+2} + a_1d^{n+1} + a_0d^n = 0.$$

Adding up the other equations, we obtain

$$g_{n+3} + a_2g_{n+2} + a_1g_{n+1} + a_0g_n = 0.$$

Given the equation  $g_3 + a_2g_2 + a_1g_1 + a_0$  we conclude  $g_3$  is real. Then by induction on  $n$ , it follows that all  $g_n$  are real.

**Solution 18:** First notice  $n \sin n + (180 - n) \sin(180 - n) = 180 \sin n$ .

Thus, we want the value of  $2(\sin 2 + \sin 4 + \dots + \sin 88) + \sin 90$ . (I divided by 90 because it asked about the average).

We need to show given  $w = cis2$  that

$$-i(w + w^2 + \dots + w^{44} - \frac{1}{w} - \frac{1}{w^2} - \dots - \frac{1}{w^{44}}) + 1 = i(\frac{w+1}{w-1}).$$

This is equivalent to

$$w + w^2 + \dots + w^{44} + w^{46} + w^{47} + \dots + w^{89} + i = -\frac{w+1}{w-1}.$$

Since  $i = w^{45}$ , we may obtain that

$$w(w^{89} - 1) = -(w + 1) \text{ which is true using } w^{90} = -1.$$