DE1-MEM
Complex Numbers
Tutorial Sheet

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Tutors: Mathsmos

Self Diagnostics Checklist
If you get stuck, have you...

- Drawn a sketch?
- Used WolframAlpha Pro step by step solutions?
- Asked a GTA?
- Asked a friend?
- Gone over the notes?

Essential Questions

Problem 1.

Express the flowing complex numbers in their polar form, $r(\cos\theta + \mathrm{i}\sin\theta)$.

(a) $z = 3 + 4\mathrm{i}$

$\Rightarrow$ Find the r (the magnitude of z):
$r = |z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

Find the argument $arg(z)$ (angle θ):
$\theta = \arg(z) = \tan^{-1}\frac{imaginary\ part}{real\ part} = \tan^{-1}\frac{4}{3} = 53.13°$

$z = a + bi = r(\cos\theta + \mathrm{i}\sin\theta)$

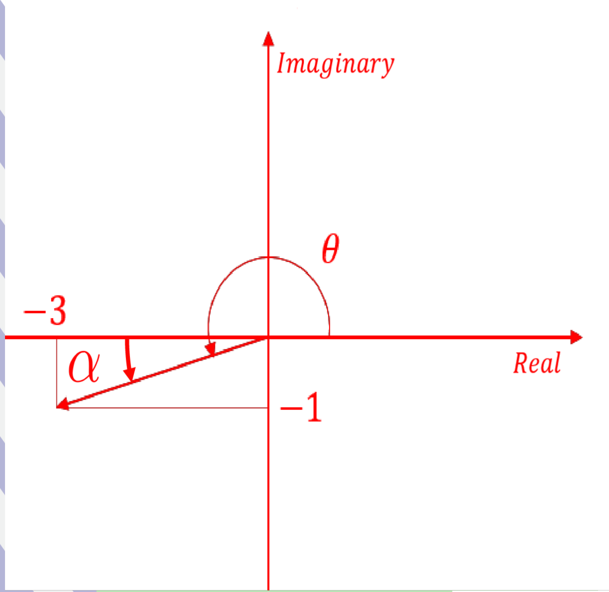
Therefore $z = 5(\cos 53.13° + \mathrm{i}\sin 53.13°)$

. . . .

(b) $z = -3 - \mathrm{i}$

$\Rightarrow \quad r = |z| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10} = 3.16$

Plot z on an Argand diagram:



$\alpha = \tan^{-1}\frac{(-1)}{(-3)} = \tan^{-1}\frac{1}{3} = 18.43°$
As a result, $arg(z) = \theta = 18.43° + 180° = 198.43°$

Therefore, in polar form:

$z = 3.16(\cos 198.43° + \mathrm{i}\sin 198.43°)$

Alternative answer: $z = 3.16(\cos -161.57° + \mathrm{i}\sin -161.57°)$

....

(c) $z = -\mathrm{i}$

$\Rightarrow \quad r = |z| = \sqrt{(0)^2 + (-1)^2} = \sqrt{1} = 1$

Plot $z = -\mathrm{i}$ on an Argand diagram:
$\theta = \arg(z) = \tan^{-1}\frac{(-1)}{(0)} + 180° = \tan^{-1}\infty + 180° = 90° + 180° = 270°.$

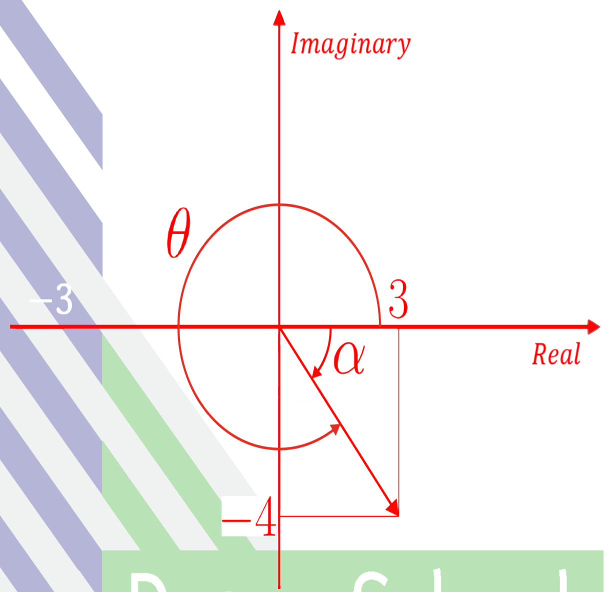
Therefore $z = 1(\cos 270° + \mathrm{i}\sin 270°)$

Alternative answer: $z = 1(\cos -90° + \mathrm{i}\sin -90°)$

....

(d) $z = 3 - 4\text{i}$

$\Rightarrow \quad r = |z| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$



$\alpha = \tan^{-1}\frac{4}{3} = 53.13$
$\theta = \arg(z) = 360° - 53.13° = 306.87°$

Therefore $z = 5(\cos 306.87° + \text{i}\sin 306.87°)$

Alternative answer: $z = 5(\cos -53.13° + \text{i}\sin -53.13°)$

Problem 2.

(I) Consider the complex numbers $z_1 = 2 + 4\text{i}$ and $z_2 = 4 - 7\text{i}$.

(a) Find $z_1 + z_2$

$\Rightarrow$ Collect real and complex terms:
$z_1 + z_2 = (2 + 4\text{i}) + (4 - 7\text{i}) = (2 + 4) + (4 - 7)\text{i} = 6 - 3\text{i}$

....

(b) Find $z_1 - z_2$

$\Rightarrow \quad z_1 - z_2 = (2 + 4\text{i}) - (4 - 7\text{i}) = (2 - 4) + (4 + 7)\text{i} = -2 + 11\text{i}$

....

(c) Find $z_1 z_2$

$\Rightarrow$ Expand the brackets:
$z_1 z_2 = (2 + 4\text{i})(4 - 7\text{i}) = 2 \times 4 - 2 \times 7\text{i} + 4 \times 4\text{i} - 4 \times 7\text{i}^2$

Collect real and complex terms:
$8 + 28 - 14\text{i} + 16\text{i} = 36 + 2\text{i}$

(d) Find $\frac{z_1}{z_2}$

$\Rightarrow$ $\frac{z_1}{z_2} = \frac{2+4\mathrm{i}}{4-7\mathrm{i}} = \frac{2+4\mathrm{i}}{4-7\mathrm{i}} \cdot \frac{4+7\mathrm{i}}{4+7\mathrm{i}} = \frac{8+16\mathrm{i}+14\mathrm{i}-28}{16+49} = \frac{-20+30\mathrm{i}}{65} = -\frac{4}{13} + \frac{6}{13}\mathrm{i}$

$\Rightarrow$ Note: Simplify a complex fraction $\frac{a+bi}{c+di}$ by multiplying the fraction with the complex conjugate of the denominator over itself (effectively multiplying by 1), i.e., $\frac{a+bi}{c+di} \cdot \frac{c-di}{c-di}$

(II)Manipulating complex numbers.

(a) Find the real and imaginary part of $z = \frac{\mathrm{i}-4}{2\mathrm{i}-3}$.

$\Rightarrow$ Simplify and collect real and complex terms:
$z = \frac{\mathrm{i}-4}{2\mathrm{i}-3} = \frac{\mathrm{i}-4}{2\mathrm{i}-3} \cdot \frac{2\mathrm{i}+3}{2\mathrm{i}+3} = \frac{-12-2+3\mathrm{i}-8\mathrm{i}}{-4-9} = \frac{14}{13} + \frac{5}{13}\mathrm{i}$.

Therefore, $\mathrm{Re}(z) = \frac{14}{13}$ and $\mathrm{Im}(z) = \frac{5}{13}$

(b) Find the **absolute value** and the **conjugate** of $z = (1+\mathrm{i})^6$

$\Rightarrow$ Express z in polar form:
$z = (1+\mathrm{i})^6 = (\sqrt{2}(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4})^6$

Using De Moivre's Theorem:
$(\sqrt{2}(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4})^6 = 8(\cos\frac{3\pi}{2} + \mathrm{i}\sin\frac{3\pi}{2}) = -8i$

Hence, $|z| = 8$ and $\overline{z} = 8\mathrm{i}$

(c) Find the **absolute value** and the **conjugate** of $w = \mathrm{i}^{17}$

$\Rightarrow$ Considering $\mathrm{i}^4 = 1$

$w = \mathrm{i}^{17} = \mathrm{i} \cdot \mathrm{i}^{16} = \mathrm{i} \cdot (\mathrm{i}^4)^4 = \mathrm{i} \cdot (1)^4 = \mathrm{i}$.

Hence,$|w| = 1$ and $\overline{w} = -\mathrm{i}$

(d) Simplify the complex number $\frac{1+\mathrm{i}}{1-\mathrm{i}} - (1+2\mathrm{i})(2+2\mathrm{i}) + \frac{3-\mathrm{i}}{1+\mathrm{i}}$

$\Rightarrow$ Evaluate each part:
$\frac{1+\mathrm{i}}{1-\mathrm{i}} = \frac{1+\mathrm{i}}{1-\mathrm{i}} \cdot \frac{1+\mathrm{i}}{1+\mathrm{i}} = \frac{1-i^2+2\mathrm{i}}{2} = \mathrm{i}$

$-(1+2\mathrm{i})(2+2\mathrm{i}) = 2-6\mathrm{i}$

$\frac{3-\mathrm{i}}{1+\mathrm{i}} = \frac{3-\mathrm{i}}{1+\mathrm{i}} \cdot \frac{1-\mathrm{i}}{1-\mathrm{i}} = \frac{3+\mathrm{i}^2-3\mathrm{i}-\mathrm{i}}{1-\mathrm{i}^2} = 1-2\mathrm{i}$

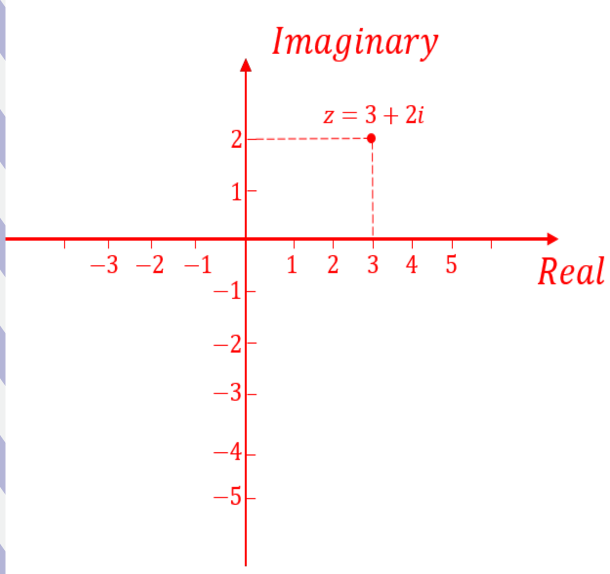
Therefore:
$\frac{1+\mathrm{i}}{1-\mathrm{i}} - (1+2\mathrm{i})(2+2\mathrm{i}) + \frac{3-\mathrm{i}}{1+\mathrm{i}} = \mathrm{i} + 2 - 6\mathrm{i} + 1 - 2\mathrm{i} = 3 - 7\mathrm{i}$

(e) Simplify the complex number $2\mathrm{i}(\mathrm{i}-1)+(\sqrt{3}-\mathrm{i})^3+(1+\mathrm{i})(1-\mathrm{i})$

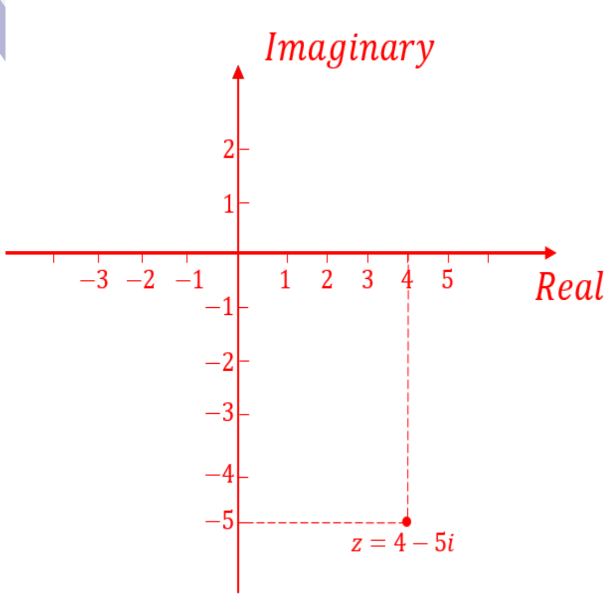
$\Rightarrow$ Expand the brackets and evaluate:
$2\mathrm{i}(\mathrm{i}-1)+(\sqrt{3}-\mathrm{i})^3+(1+\mathrm{i})(1-\mathrm{i})$
$=2\mathrm{i}^2-2\mathrm{i}+3\sqrt{3}-3\mathrm{i}(\sqrt{3})^2+3\mathrm{i}^2\sqrt{3}+(-\mathrm{i})^3+1-\mathrm{i}^2$
$=-2-2\mathrm{i}+3\sqrt{3}-3\sqrt{3}-8\mathrm{i}+2=-10\mathrm{i}$

Problem 3.

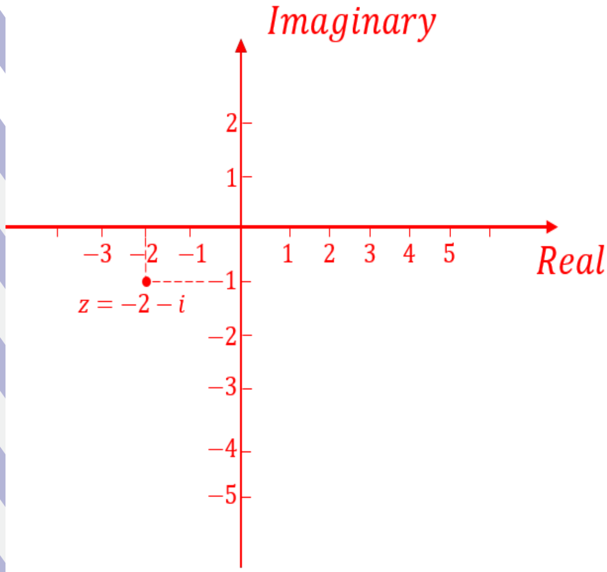
Plot the flowing complex numbers on an Argand diagram:
(a) $z=3+2\mathrm{i}$



(b) $z=4-5\mathrm{i}$



(c) $z = -2 - \mathrm{i}$



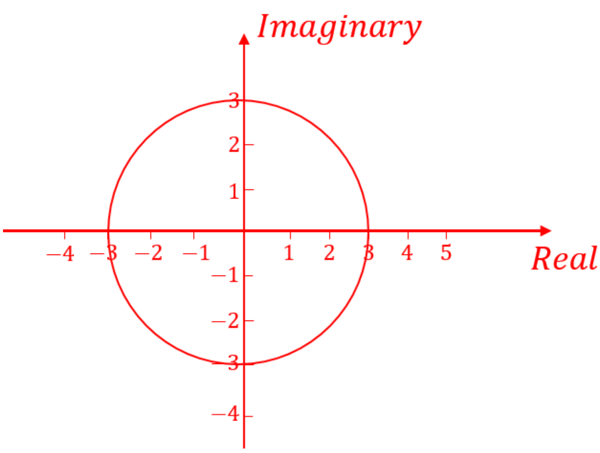
(d) $|z| = 3$

$\Rightarrow$ On an Argand diagram, plot the locus defined by $|z| = 3$.

$z = x + \mathrm{i}y$
$|z| = |x + \mathrm{i}y| = 3$
$\sqrt{x^2 + y^2} = 3$

Therefore:
$x^2 + y^2 = 9$

The solution consists of all the points lying on the circle of radius 3 with center (0,0).



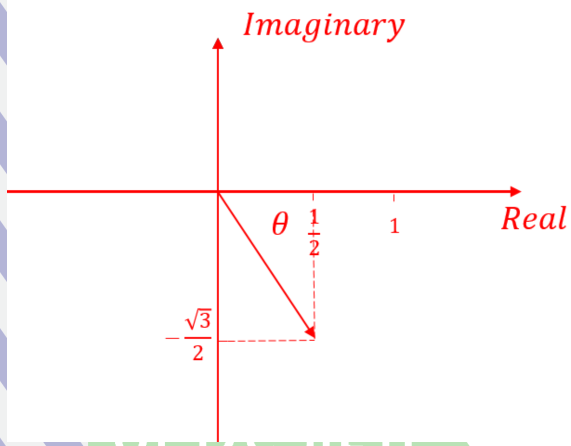
Problem 4.

Write the following complex number in polar and exponential forms

$z = \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}$

$\Rightarrow \quad |z| = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1.$

From diagram below:
$\theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$



Therefore $\arg(z) = -\frac{\pi}{3}$
or alternatively: $\arg(z) = \frac{5\pi}{3}$

Complex number z in polar form:
$z = \cos\frac{-\pi}{3} + \mathrm{i}\sin\frac{-\pi}{3} = \cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}$
or: $\cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3}$

In exponential form: $e^{-\frac{\pi}{3}\mathrm{i}}$
or: $e^{\frac{5\pi}{3}\mathrm{i}}$

Problem 5.

Application of de Moivres theorem.

(a) Find $(\cos\theta + \mathrm{i}\sin\theta)^{-10}$ in the form $(\cos(A\theta) - \mathrm{i}\sin(B\theta))$

$\Rightarrow$ Using de Moivres theorem:
$(\cos\theta + \mathrm{i}\sin\theta)^{-10} = (\cos(-10\theta) + \mathrm{i}\sin(-10\theta)) = (\cos(10\theta) - \mathrm{i}\sin(10\theta))$

. . . .

(b) Simplify the flowing expression: $\frac{\cos 2\theta + \mathrm{i}\sin 2\theta}{\cos 3\theta + \mathrm{i}\sin 3\theta}$

$\Rightarrow$ Using de Moivres theorem:
$\frac{\cos 2\theta + \mathrm{i}\sin 2\theta}{\cos 3\theta + \mathrm{i}\sin 3\theta} = \frac{(\cos\theta + \mathrm{i}\sin\theta)^2}{(\cos\theta + \mathrm{i}\sin\theta)^3}$

Simplify:
$\frac{(\cos\theta + \mathrm{i}\sin\theta)^2}{(\cos\theta + \mathrm{i}\sin\theta)^3} = \frac{(\cos\theta + \mathrm{i}\sin\theta)^2}{(\cos\theta + \mathrm{i}\sin\theta)^2(\cos\theta + \mathrm{i}\sin\theta)} = \frac{1}{\cos\theta + \mathrm{i}\sin\theta}(\frac{\cos\theta - \mathrm{i}\sin\theta}{\cos\theta - \mathrm{i}\sin\theta}) = \cos\theta - \mathrm{i}\sin\theta$

. . . .

(c) Prove that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\Rightarrow$ Consider the complex number $\cos 3\theta + \mathrm{i}\sin 3\theta$

By de Moivres theorem:
$\cos 3\theta + \mathrm{i}\sin 3\theta = (\cos\theta + \mathrm{i}\sin\theta)^3$

$\cos 3\theta + \mathrm{i}\sin 3\theta = \cos^3\theta + 3\cos^2\theta(\mathrm{i}\sin\theta) + 3\cos\theta(\mathrm{i}\sin\theta)^2 + (\mathrm{i}\sin\theta)^3$
$\cos 3\theta + \mathrm{i}\sin 3\theta = \cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta$
$\cos 3\theta + \mathrm{i}\sin 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta + \mathrm{i}(3\cos^2\theta\sin\theta - \sin^3\theta)$

Comparing real parts of each side of the equation above, you obtain:
$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

Exam Style Questions

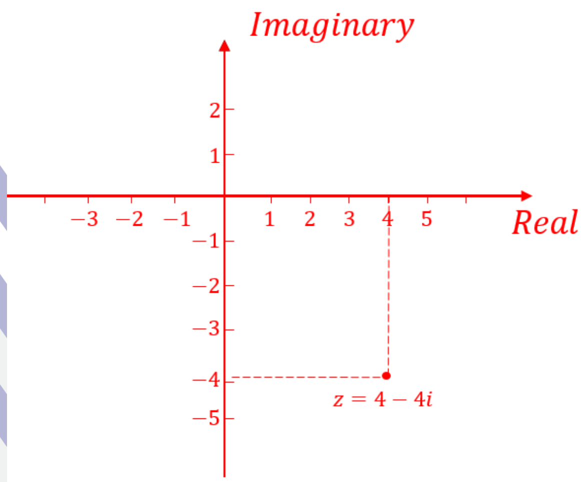
Problem 6.

a) Write down the modulus and argument of the complex number $4 - 4\mathrm{i}$. Solve the equation $z^5 = 4 - 4\mathrm{i}$, expressing your answers in the exponential form.

b) State the solution, to part a, with the smallest positive argument and find the real part of it (in polar form).

$\Rightarrow$ a) $|z^5| = |4 - 4\mathrm{i}| = \sqrt{(4^2 + (-4)^2)} = 4\sqrt{2}$

Quick sketch of z^5 on an Argand diagram:



From the Argand diagram above:
$\arg(z^5) = \arg(4 - 4\mathrm{i}) = -\frac{\pi}{4}$
or, $\arg(z^5) = \frac{7\pi}{4}$

Therefore, $4 - 4\mathrm{i} = 4\sqrt{2}(\cos\frac{\pi}{4} - \mathrm{i}\sin\frac{\pi}{4})$
or, $4 - 4\mathrm{i} = 4\sqrt{2}(\cos\frac{7\pi}{4} + \mathrm{i}\sin\frac{7\pi}{4})$

Rewrite the argument for the complex number, $4 - 4\mathrm{i}$, in its general from: $2n\pi - \frac{\pi}{4}$, where n is an integer.
Note: For any integer n, $\cos(2n\pi - \frac{\pi}{4}) = \cos(-\frac{\pi}{4})$. Likewise for $\sin(-\frac{\pi}{4})$.

Now, $z^5 = \cos(2\pi - \frac{\pi}{4}) + \mathrm{i}\sin(2\pi - \frac{\pi}{4})$

Model the solutions, z_n, to $z^5 = 4 - 4\mathrm{i}$ as complex numbers in polar form, i.e.:
$z_n = r(\cos\theta + \mathrm{i}\sin\theta)$

If $z_n = r(\cos\theta + \mathrm{i}\sin\theta)$, then by de Moivres theorem: $z^5 = r^5(\cos 5\theta + \mathrm{i}\sin 5\theta)$

$r^5(\cos 5\theta + \mathrm{i}\sin 5\theta) = 4\sqrt{2}(\cos\frac{\pi}{4} - \mathrm{i}\sin\frac{\pi}{4})$

Compare magnitudes:
$r^5 = 4\sqrt{2}, r = \sqrt{2}$

Compare arguments:
$5\theta = 2n\pi - \frac{\pi}{4}, \theta = (8n - 1)\frac{\pi}{20}$

For appropriate values of n, so that θ lies between $-\pi$ and π:

$$n = -2 \quad \Rightarrow \quad \theta = \frac{-17\pi}{20}$$

$$n = -1 \quad \Rightarrow \quad \theta = \tfrac{-9\pi}{20}$$

$$n = 0 \quad \Rightarrow \quad \theta = \tfrac{-\pi}{20}$$

$$n = 1 \quad \Rightarrow \quad \theta = \tfrac{7\pi}{20}$$

$$n = 2 \quad \Rightarrow \quad \theta = \tfrac{15\pi}{20} \text{ or } \tfrac{3\pi}{4}$$

Therefore solutions in exponential are:

$$z_1 = \sqrt{2}e^{\frac{-17\pi}{20}i},\ z_2 = \sqrt{2}e^{\frac{-9\pi}{20}i},\ z_3 = \sqrt{2}e^{\frac{-\pi}{20}i},\ z_4 = \sqrt{2}e^{\frac{7\pi}{20}i}, \text{ and } z_5 = \sqrt{2}e^{\frac{3\pi}{4}i}$$

$\Rightarrow$ b) Solution with the smallest positive argument: $\sqrt{2}e^{\frac{7\pi}{20}i}$

$Re\{\sqrt{2}e^{\frac{7\pi}{20}i}\} = \sqrt{2}\cos(\frac{7\pi}{20})$