Integral of Inverse Exponential Logarithmic Function from Zero to Infinity

Ivan V. Morozov

August 2019

Suppose one has the integral

$$\int_0^\infty \frac{1}{x^{\ln x}}\,dx$$

How would it be integrated over the given interval? There is a property which allows for a function to be expressed as the constant e raised to the power of the natural logarithm of that function,

$$u = e^{\ln u}$$

This can be applied to the very same integrand in the integral $\int_0^\infty \frac{1}{x^{\ln x}}\,dx$. $\frac{1}{x^{\ln x}}$ can be expressed as $x^{-\ln x}$, and using the exponential property, it is shown to be that

$$x^{-\ln x} = e^{\ln(x^{-\ln x})}$$

Using the property of the natural log function

$$\ln(a^b) = b\ln a$$

the function $e^{\ln(x^{-\ln x})}$ can be expressed as such,

$$x^{-\ln x} = e^{-\ln x\,\ln x}$$

or simply

$$e^{-\ln^2 x}$$

Taking the original integral $\int_0^\infty \frac{1}{x^{\ln x}}\,dx$ and substituting the integrand with the derived expression, the integral can be represented as

$$\int_0^\infty e^{-\ln^2 x}\,dx$$

This problem can now be solved with integration by parts. One can substitute $\ln x$ for arbitrary variable u such that

$$u = \ln x$$

Therefore $du = \frac{1}{x}$ for the logarithmic derivative of x is $\frac{1}{x}$, and it can be said that

$$dx = x\,du$$

$$\ln^2 x = u^2$$

and

$$x = e^u$$

From this, the original integral can be shown to be

$$\int_0^\infty e^{u-u^2}\,du$$

Taking the expression $u - u^2$, one can complete the square to get the following,

$$u - u^2$$

$$= -(u^2 - u)$$

$$= \frac{1}{4} - (u^2 - u + \frac{1}{4})$$

$$= \frac{1}{4} - (u - \frac{1}{2})^2$$

Substituting this into the integral gives

$$\int_0^\infty e^{\frac{1}{4}-(u-\frac{1}{2})^2}\,du$$

Another substitution can be applied

$$v = u - \frac{1}{2}$$

and therefore $dv = 1$ since $\frac{d}{dx}\,u = 1$ and $\frac{d}{dx}\,\frac{1}{2} = 0$. It can be concluded that $dv = du$, and substituting v into the integral, one has

$$\int_0^\infty e^{\frac{1}{4}-v^2}\,dv$$

and when $e^{\frac{1}{4}}$ is factored out,

$$\sqrt[4]{e}\int_0^\infty e^{-v^2}\,dv$$

A factor of of $\frac{\sqrt{\pi}}{2}$ can be added to the outside of the integral and its reciprocal $\frac{2}{\sqrt{\pi}}$ to its inside. Now the integral becomes

$$\frac{\sqrt[4]{e}\sqrt{\pi}}{2}\int_0^\infty \frac{2}{\sqrt{\pi}}e^{-v^2}\,dv$$

The Gauss error function $\operatorname{erf}(v)$ is defined as

$$\operatorname{erf}(v) = \frac{2}{\sqrt{\pi}} \int_0^v e^{-t^2}\, dt$$

Or as an indefinite integral form,

$$\operatorname{erf}(v) = \frac{2}{\sqrt{\pi}} \int e^{-v^2}\, dv$$

This definition can be substituted into the previously derived equation to become

$$\left[\frac{\sqrt[4]{e}\sqrt{\pi}\operatorname{erf}(v)}{2}\right]_0^\infty$$

The substitution $v = u - \frac{1}{2}$ can be undone to get

$$\left[\frac{\sqrt[4]{e}\sqrt{\pi}\operatorname{erf}(u - \frac{1}{2})}{2}\right]_0^\infty$$

and since it was stated earlier that $u = \ln x$, u also can be undone and substituted back in to the equation,

$$\left[\frac{\sqrt[4]{e}\sqrt{\pi}\operatorname{erf}(\ln x - \frac{1}{2})}{2}\right]_0^\infty$$

This can be simplified to

$$\frac{\sqrt[4]{e}\sqrt{\pi}\operatorname{erf}(\ln\infty - \frac{1}{2})}{2} - \frac{\sqrt[4]{e}\sqrt{\pi}\operatorname{erf}(\ln 0 - \frac{1}{2})}{2}$$

and further simplified to

$$\frac{\sqrt[4]{e}\sqrt{\pi}\operatorname{erf}(\infty)}{2} - \frac{\sqrt[4]{e}\sqrt{\pi}\operatorname{erf}(-\infty)}{2}$$

The limit of the Gauss error function $\operatorname{erf}(x)$ as x approaches infinity, is

$$\lim_{x\to\infty} \operatorname{erf}(x) = 1$$

and the limit of that function as x approaches negative infinity equals

$$\lim_{x\to-\infty} \operatorname{erf}(x) = -1$$

Therefore, $\operatorname{erf}(\infty) = 1$ and $\operatorname{erf}(-\infty) = -1$, and plugging this into the equation, it becomes

$$\frac{\sqrt[4]{e}\sqrt{\pi}}{2} + \frac{\sqrt[4]{e}\sqrt{\pi}}{2}$$

which simplifies to

$$\sqrt[4]{e}\sqrt{\pi}$$

From this it can be said that

$$\int_0^\infty \frac{1}{x^{\ln x}}\, dx = \sqrt[4]{e}\sqrt{\pi}$$