Lambert W's Taylor Series

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The Lambert W function is defined as the inverse of xe^x. That is:

$$y = W(x) \iff x = ye^y$$

It turns out that this has a nice Taylor series:

$$W(x) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} x^k$$

We will derive this, and we'll take a slightly unusual path to get there.

Taylor's theorem is:

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(a) \frac{(x-a)^k}{k!}$$

I could use this theorem directly on $W(x)$, but that involves differentiating $W(x)$ a bunch of times and seeing if I can find a pattern. That's really messy. I'll use a more interesting approach.

In fact, I'll only need to use this theorem on polynomials. This avoids issues of convergence; for polynomials, the Taylor series is really a finite sum, because if k is large enough, then $f^{(k)} = 0$. (Also, Taylor's theorem is much easier to prove for polynomials than for general functions.)

Let's make a useful change of notation. Instead of writing f' for the derivative of f, let's write Df. (Here, D is an *operator* – it turns a function into a function.) Additionally, $\frac{x^k}{k!}$ is *such* an important polynomial that I'll give it a special name: $d_k(x) := \frac{x^k}{k!}$. Note that:

- $Dd_k = d_{k-1}$
- $d_k(0) = 0$ (when $k \neq 0$)
- $d_0 = 1$

d_k is called the *basic sequence* of D.

Our revised Taylor series looks like:

$$f(x) = \sum_{k=0}^{\infty} (D^k f)(a)\ d_k(x-a)$$

We can add together operators. For example, $D + D^2$ is the operator such that $(D + D^2)f = f' + f''$. I is the identity operator, i.e., that $If = f$ for every function f. We have $D^0 = I$.

In addition, we can do weird things such as find e^D – since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, we can define e^D to mean $\sum_{k=0}^{\infty} \frac{D^k}{k!}$.

Define the operator E as follows: $(Ef)(x) = f(x+1)$. That is, E shifts f over one. More generally, $(E^a f)(x) = f(x + a)$. A nice fact is that $DE = ED$ (that is, they commute). We will now prove that $E = e^D$.

Start with the Taylor series, and substitute $x \mapsto x + 1$ and $a \mapsto x$:

$$\begin{aligned}
f(x+1) &= \sum_{k=0}^{\infty} (D^k f)(x)\ d_k(x+1-x) \\
f(x+1) &= \sum_{k=0}^{\infty} (D^k f)(x)\ d_k(1) \\
(Ef)(x) &= \sum_{k=0}^{\infty} \frac{(D^k f)(x)}{k!} \\
E &= \sum_{k=0}^{\infty} \frac{D^k}{k!} \\
E &= e^D
\end{aligned}$$

Define the *Abel operator* $A := DE$. That is, $(Af)(x) = f'(x + 1)$. By the above theorem, $A = De^D$. We have A written in terms of D. Can we express D in terms of A? That is, can we find the coefficients c_k of the series:

$$D = \sum_{k=0}^{\infty} c_k A^k$$

And this is where this ties into the Lambert W function. Since $W(x)$ is the inverse of xe^x, and $A = De^D$, we have $D = W(A)$. That means the same coefficients c_k will be the coefficients of the series for $W(x)$.

This is a variant of Taylor's theorem, and is equally true:

$$f(x) = \sum_{k=0}^{\infty} (A^k f)(a)\ a_k(x-a)$$

where a_k is the basic sequence for A – that is, $Aa_k = a_{k-1}$, $a_k(0) = 0$ when $k \neq 0$, and $a_0 = 1$. We will figure out what the a_k are later. Basically, this is the Taylor sequence with all of the Ds replaced by As. Again, f only needs to be a polynomial. (The proof of this is similar to how you'd prove Taylor's theorem for polynomials.)

Differentiating:

$$(Df)(x) = \sum_{k=0}^{\infty} (A^k f)(a)\ a'_k(x-a)$$

Set $a = x$:

$$(Df)(x) = \sum_{k=0}^{\infty} (A^k f)(x)\ a'_k(0)$$

$$(Df)(x) = \sum_{k=0}^{\infty}\ a'_k(0)\ (A^k f)(x)$$

$$D = \sum_{k=0}^{\infty} a'_k(0) A^k$$

This means that the coefficients of the Lambert W function are precisely $a'_k(0)$, where a_k is the basic sequence of A !

So, what are the a_k? Let's list the first few and see if we find a pattern. Remember that $A = DE$. Also, $Aa_k = a_{k-1}$, $a_k(0) = 0$ when $k \neq 0$, and $a_0 = 1$. So:

- $a_0(x) = 1$
- $a_1(x) = (x-1) + 1$

  I'm writing this in a slightly weird way. Think of it as me doing A backwards, by integrating a_0 and then shifting it. We have $Aa_1 = a_0$. The 1 is to ensure that $a_1(0) = 0$.

- $a_2(x) = \dfrac{(x-2)^2}{2} + (x-2)$

  It's easy to check that $Aa_2 = a_1$. We have $a_2(0) = \frac{4}{2} - 2 = 0$.

- $a_3(x) = \dfrac{(x-3)^3}{3!} + \dfrac{(x-3)^2}{2!}$

  We check that $a_3(0) = -\frac{27}{6} + \frac{9}{2} = 0$

Generalizing the pattern, we have:

$$a_k(x) = \frac{(x-k)^k}{k!} + \frac{(x-k)^{k-1}}{(k-1)!}$$

(except for $k = 0$, where $a_k = 1$). The three conditions for a_k are satisfied, as you can check.

Now, all we need to do is compute $a'_k(0)$:

$$\begin{aligned}
a'_k(x) &= \frac{(x-k)^{k-1}}{(k-1)!} + \frac{(x-k)^{k-2}}{(k-2)!} \\
&= \left(\frac{(x-k)^{k-2}}{(k-1)!}\right)((x-k) + (k-1)) \\
&= \left(\frac{(x-k)^{k-2}}{(k-1)!}\right)(x-1) \\
a'_k(0) &= \frac{(-k)^{k-2}}{(k-1)!}(-1) \\
&= \frac{(-k)^{k-2}}{(k-1)!}\frac{(-k)}{k} \\
&= \frac{(-k)^{k-1}}{k!}
\end{aligned}$$

(except for $k = 0$, where $a'_k(0) = 0$).

That means that, by our above result:

$$D = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} A^k$$

and, thus:

$$W(x) = \sum_{k=1}^{\infty} \frac{(-k)^{k-1}}{k!} x^k$$

And we are done.