The Representation Theory of Groups

Project Presentation

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Contents

Introduction

- Aim of project: A self-study of the representation theory of groups
- Focus has been on the case where the group is finite
- The infinite (particularly compact) case has been touched upon, but full understanding requires a knowledge of measure theory

The Finite case: concepts covered

I learnt about:

1. The basic definitions: Representations, Subrepresentations, Intertwining Operators, Isomorphisms, Irreducibility, complete reducibility, indecomposability, Faithfulness, and unitarity
2. Character Theory: The definition and some key properties
3. Key theorems: Complete reducibility for finite dimensional representations, Uniqueness of such decompositions (up to isomorphism on the irreducible representations), orthogonality of characters
4. McKay graphs

The Finite case: What I did I

I have looked at:

1. Showing that S_3 is isomorphic to $\langle g, r : g^3 = r^2 = 1, gr = rg^{-1} \rangle$ and so any representation of S_3 is determined by the action of the associated homomorphism is determined by its action on g and r
2. Decomposing of the tensor products of the irreducible representations of S_3 by finding isomorphisms
3. The same but using character theory
4. Decomposing the Permutation Representation of S_3
5. Finding the Mckay Graphs for the irreducible representations of S_3
6. Showing that if V is an irreducible representation of some group then the complex conjugate representation $\overline{V}$ is also irreducible.

The Finite case: What I did II

7. Showing that if an irreducible representation is isomorphic to its complex conjugate representation then the McKay graph for that representation is undirected.
8. Showing that one of the irreducible representations of S_3 is faithful

The Finite case: The character table for S_3

The Compact case: Concepts covered

As mentioned, a thorough treatment of this material would require measure theory; nevertheless I am aware of the following:

1. The inner product now requires integrating over the group rather than averaging, which is where measure theory is required
2. Complete Reducibility still holds
3. Orthogonality of Characters still holds
4. Due to the group being infinite and compact, results from areas such as Analysis and Topology are of importance

The Compact case: what I did

For the compact case, I considered:

1. The definition of $SU(2)$ and showed that it is equivalent to $\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\}$
2. The decomposition of the tensor product representations of $SU(2)$, using a referenced formula for the irreducible representations
3. Some of the McKay graphs for the irreducible representations of $SU(2)$ and showed that every McKay graph for the irreducibles of $SU(2)$ are non-directed as each irreducible is isomorphic to its complex conjugate representation (which holds as in the finite case).

DETAILS

Avenues for further investigation: Faithful Representations

Faithful representations are particularly interesting as if $GL(V)$ is restricted to $Im(\rho)$ (V is a faithful representation with associated homomorphism ρ), an inverse homomorphism is defined and the homomorphism is an isomorphism.

Some questions worth asking then are:

1. Does every group have at least 1 faithful irreducible representation?
2. What are some, easy to check, necessary conditions and sufficient conditions for faithful representations?

Avenues for further investigation: Finding irreducible representations

I emphasised, in the report, the power of character theory for finding complete decompositions of representations once the irreducible representations are known. An important process to consider, is how to find such representations.

Avenues for further investigation: Other thoughts

Some other thoughts are:

1. How do the results shown in the report change if the field over which the representation is defined is changed?
2. What are some Scientific applications of Representation theory?
3. This project looked at S_3 and $SU(2)$, so the natural extension is to consider S_n and $SU(n)$