THE VOLUME OF n-BALLS

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Introduction

- Our goal is to derive a formula for the volume of n-dimensional balls in $\mathbb{R}^n$.
- Let's begin with some familiar definitions, and we will rely on our intuition to start.

Definition

For a natural number $n \geq 1$, an $(n-1)$-dimensional sphere of radius r is the set of all points in $\mathbb{R}^n$ which are a fixed distance r from a given center point.

- We take the center to be the origin and denote the $(n-1)$-sphere of radius r in $\mathbb{R}^n$ by $\mathbb{S}^{n-1}(r)$. That is,

$$\mathbb{S}^{n-1}(r) = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + ... + x_n^2 = r^2\}$$

- First, notice when $n = 1$, the 0-sphere is just the two points on the real line at r and $-r$.
- When $n = 2$, we have $\mathbb{S}^1(r) = \{(x_1, x_2) \in \mathbb{R}^2 \ \mid x_1^2 + x_2^2 = r^2\}$
- Taking $n = 3$, $\mathbb{S}^2(r)$ is the sphere in $\mathbb{R}^3$ given by $\mathbb{S}^2(r) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = r^2\}$
- Now, $\mathbb{S}^n(r)$ is harder to visualize in higher dimensions, but we can use our intuition of lower dimensional spheres to help us.

- Let's try and visualize the 3-sphere. If we take a 0-sphere, which is just the endpoints of a line segment in $\mathbb{R}^1$, and rotate it about the origin, what do we have?

Correct. We have a 1-sphere in $\mathbb{R}^2$, also known as a circle.

- Now, if we take this circle and rotate every point about any axis going through the center point and lying in the $\mathbb{R}^2$ plane, we will have the 2-sphere in $\mathbb{R}^3$.
- We think of the 3-sphere in the same way. If we take the 2-sphere in $\mathbb{R}^3$, and rotate every point about any axis going through the center point, we will have the 3-sphere in $\mathbb{R}^4$.
- This is difficult to visualize, but this inductive process we are doing in our minds is actually exactly what we will do mathematically.
- So.....let the fun begin :)

Recall:

Orthogonal matrices represent linear transformations that preserve the dot product of vectors. They represent isometries of Euclidean space (distance preserving) and denote rotations or reflections.

- By definition, orthogonal matrices have determinant ± 1. The matrices in the group of orthogonal matrices in $\mathbb{R}^n$ with determinant $+1$ represent the rotations. These are called special orthogonal matrices and are given by

$\mathrm{SO}(n) = \{A : A^T A = I; detA = 1\}$

Consider the following rotation given as a square matrix in $\mathrm{SO}(n+1)$.

$$A_j = \begin{bmatrix} I_{j-1} & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & I_{n-j} \end{bmatrix}$$

for $1 \leq j \leq n$, where

$$R = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

is a 2×2(counter-clockwise) rotation matrix, I_k is the $k \times k$ identity matrix, and j specifies where the rotation matrix is placed.

For example, A_1 is the $(n+1) \times (n+1)$ matrix

$$A_1 = \begin{bmatrix} R & 0 \\ 0 & I_{n-1} \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Notice that in this case and in general, the determinant will always be 1.
- Also, since $A_j^T = A_j^{-1}$, A_j is in the special orthogonal group.

Okay cool. So how does this help us?

- To help us see that these matrices generate spheres in $\mathbb{R}^{n+1}$, let's look at the case with $n = 3$ to find a parametrization of a 3-sphere in $\mathbb{R}^4$. We start with the point $P = (1, 0, 0, 0)$ in $\mathbb{R}^4$ and inductively apply our rotations. Applying the rotation A_1 to P for $0 \leq \theta < 2\pi$, we have

$$\begin{bmatrix} cos\theta & -sin\theta & 0 & 0 \\ sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = (cos\theta, sin\theta, 0, 0)$$

- Notice that this is a parametrization of the circle $S^1 \subset \mathbb{R}^4$ lying in the x_1x_2-plane.

Now let's apply the rotation A_2 to our circle in the x_1x_2-plane. This gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\phi & -sin\phi & 0 \\ 0 & sin\phi & cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta \\ sin\theta \\ 0 \\ 0 \end{bmatrix} = (cos\theta, cos\phi sin\theta, sin\phi sin\theta, 0)$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. This gives us a 2-sphere lying in $x_1x_2x_3$-space. Notice that the parametrization resembles spherical coordinates.

Continuing in this way, let the new variable ψ range from 0 to 2π, and letting ϕ and θ range from 0 to π, we have our parametrization of the 3-sphere in $\mathbb{R}^4$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & cos\psi & -sin\psi \\ 0 & 0 & sin\psi & cos\psi \end{bmatrix} \begin{bmatrix} cos\theta \\ cos\phi sin\theta \\ sin\phi sin\theta \\ 0 \end{bmatrix} = (cos\theta, cos\phi sin\theta, cos\psi sin\phi sin\theta, sin\psi sin$$

- "Thus we can see that rotations in higher dimensions can be realized as the action of a linear transformation in which there is one free parameter. This parameter does a rotation in two dimensions and leaves all other dimensions fixed."

Continuing in this way, we now have a parametrization of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ given by

$$x_1 = cos\theta_1$$

$$x_2 = sin\theta_1 cos\theta_2$$

$$x_3 = sin\theta_1 sin\theta_2 cos\theta_3$$

$$\vdots$$

$$x_{n-1} = sin\theta_1 \dots sin\theta_{n-2} cos\theta_{n-1}$$

$$x_n = sin\theta_1 \dots sin\theta_n - 2sin\theta_{n-1},$$

where $0 \leq \theta_{n-1} < 2\pi$ and $0 \leq \theta_i \leq \pi$, for $i = 1, 2, \dots, n-2$. This will be useful, I promise.

The following integral formula for $\int sin^m\theta d\theta$ will help us. For any integer $m \geq 2$, we have

$$\int_0^\pi sin^m\theta = -\frac{sin^{m-1}\theta cos\theta}{m}\Big|_{\theta=0}^{\theta=\pi} + \int_0^\pi sin^{m-2}\theta d\theta = \int_0^\pi sin^{m-2}\theta d\theta$$

Note that when m is even, say $m = 2k$, then

$$\int_0^\pi sin^{2k}\theta d\theta = \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \pi$$

Similarly, when m is odd, say $m = 2k+1$, then

$$\int_0^\pi sin^{2k+1}\theta d\theta = \frac{2k}{2k+1} \cdot \frac{2k-2}{2k-1} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 2$$

Tables and Figures

- Use `tabular` for basic tables — see Table 1, for example.
- You can upload a figure (JPEG, PNG or PDF) using the files menu.
- To include it in your document, use the `includegraphics` command (see the comment below in the source code).

Item	Quantity
Widgets	42
Gadgets	13

Table 1: An example table.

Readable Mathematics

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with $\mathrm{E}[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2 < \infty$, and let

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{n} = \frac{1}{n} \sum_{i}^{n} X_i$$

denote their mean. Then as n approaches infinity, the random variables $\sqrt{n}(S_n - \mu)$ converge in distribution to a normal $\mathcal{N}(0, \sigma^2)$.