Homework Set 1: 141 Redux

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1. Prove that:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Using Proof by Induction.

First, prove that for some n, this equation holds true.

$$\begin{aligned} n = 21^2 + 2^2 &= \frac{2(2+1)(4+1)}{6} \\ 2 + 4 &= \frac{2(3)(5)}{6} \\ 6 &= \frac{30}{6} \\ 6 &= 6 \end{aligned}$$

Now, prove that this works for any n+1.

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\ 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

Notice that the n+1 equation contains the n equation.

$$\begin{aligned} \boxed{1^2 + 2^2 + \cdots + n^2} + (n+1)^2 &= \frac{(n+1)(n+2)(2n+3)}{6} \\ \frac{n(n+1)(2n+1)}{6} + (n+1)^2 &= \frac{(n+1)(n+2)(2n+3)}{6} \\ n(n+1)(2n+1) + 6(n+1)^2 &= (n+1)(n+2)(2n+3) \\ n(2n+1) + 6(n+1) &= (n+2)(2n+3) \\ 2n^2 + n + 6n + 6 &= 2n^2 + 4n + 3n + 6 \\ 2n^2 + 7n + 6 &= 2n^2 + 7n + 6 \end{aligned}$$

therefore

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

2. Prove that

$$6 \mid n^3 - n$$

Using Proof by Induction.

First prove that this equation is valid for an arbitrary n.

$$\begin{aligned}
&n = 2 \\
&6 \mid 2^3 - 2 \\
&6 \mid 6
\end{aligned}$$

Now, prove for any n+1

$$\begin{aligned}
&6 \mid (n+1)^3 - (n+1) \\
&6 \mid n^3 + 3n^2 + 3n + 1 - n - 1 \\
&6 \mid n^3 + 3n^2 + 3n - n
\end{aligned}$$

I can pull the original equation out of this one

$$6 \mid \boxed{n^3 - n} + 3n^2 + 3n$$

Now I need to prove that $3n^2 + 3n$ is divisible by 6

$$\begin{aligned}
&6 \mid 3n^2 + 3n \\
&6 \mid 3(n^2 + n) \\
&2 \mid n^2 + n
\end{aligned}$$

Now I need to prove that $n^2 + n$ is divisible by 2, or even

Let n be even. By our proof in class today (seen in one form in problem 3), n^2 is even when n is even. An even number added to an even number is even.

Let n be odd. By the same proof, n^2 is odd when n is odd. An odd number added to an odd number is an even number. Therefore, $n^2 + n$ is an even number.

Therefore, $6 \mid (n+1)^3 - (n+1)$.

3. Prove that $\sqrt[3]{2}$ is an irrational number

   Assume that $\sqrt[3]{2}$ is a rational number. If so, then

$$\sqrt[3]{2} = \frac{a}{b}! = 0$$

   where a,b have no common factors

$$a^3 = b\sqrt[3]{2}$$
$$a^3 = 2b^3$$

   We now know that a^3 is even. It would be helpful is a was even.

   Let a^3 be even, prove that a is even

$$a^3 = 2k$$
$$a = \sqrt[3]{2k}$$
$$a = \sqrt[3]{2}\sqrt[3]{k}$$

   Blech...lets try again with the contrapositive.

   Assume that a is odd, prove a^3 is odd.

$$a = 2k + 1$$
$$a^3 = 8k^3 + 12k^2 + 6k + 1$$
$$a^3 = 2(4k^3 + 6k^2 + 3k) + 1$$

   a^3 is odd, therefore if a^3 is even, a is even. Now back.

$$a^3 = 2b^3$$
$$(2L)^3 = 2b^3$$
$$8L^3 = 2b^3$$
$$4L^3 = b^3$$
$$2(2L^3) = b^3$$

   By the same proof as above, b must be even because b^3 is even. Now, a and b share the common factor of two, therefore, $\sqrt[3]{2}$ is not rational, and therefore irrational.

4. Given G(V,E), we know that $\sum_1^n \mid d_i = 2e$. Prove

$$e \leq \frac{n(n-1)}{2}$$

Base case $n = 2$, a graph with two vertices has 1 edge.

$$e = 1$$
$$e \leq \frac{2(2-1)}{2}$$
$$1 \leq 1$$

Now prove the $n+1$ option. When the $n+1$ vertice is added, it can add up to n edges, one for each of the existing vertices.

$$e + n \leq \frac{(n+1)n}{2}$$
$$e + n \leq \frac{n^2+n}{2}$$
$$e + n \leq \frac{n^2+n+n-n}{2}$$
$$e + n \leq \frac{n^2-n}{2} + \frac{2n}{2}$$
$$e + n \leq \boxed{\frac{n(n-1)}{2}} + n$$

Therefore, by induction:

$$e \leq \frac{n(n-1)}{2}$$

5. Show that every graph with two or more nodes contains two nodes that have equal degrees.

Let us try to prove that every graph with two or more nodes have unique degrees. We know that the set of possible degrees for a graph with n vertices is:

$$0, 1, \ldots, n-1$$

This gives us a total of n unique degrees to assign to our n vertices. We must assign a degree of zero to one vertex. A vertex with degree zero is connected to no other vertices. Let us now assign the degree $n-1$ to a vertice. This vertice is connected to every other vertice in the graph. This is a contradiction, because it is impossible to simulatenously have a vertice that is connected to every other vertice, and a vertice that is connected to none. Therefore, there are at least two vertices with the same degree in any graph with at least 2 vertices.